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AMERICAN
JOURNAL OF MATHEMATICS

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ABRAHAM COHEN
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E. T. BELL

F. D. MURNAGHAN

W. A. MANNING

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume L, Number 4

OCTOBER, 1928

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

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THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $6.00 (foreign postage 25 cents); single numbers $1.50.

A few complete sets of the JOURNAL remain on sale.

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Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

E. Goursat

UNIV. OF MICH.

Affinely Connected Function Space Manifolds.

By Aristotle D. Michal.

Introduction.

The year 1887 marked the beginning of systematic research by Volterra along the line of *functions of curves or functionals.** At approximately the same time Ricci † began his systematic investigations on the *absolute calculus* by taking as a starting point the researches of Gauss, Riemann and Christoffel. The advent of Einstein's General Relativity Theory gave impetus to the study of differential invariants and differential geometry. The researches of Levi-Civita, Weyl, Veblen, Eisenhart, Cartan, Schouten and of numerous other authors are well known in this relativistic age.‡

The object of this paper is to develop a theory of affinely connected manifolds in the function space of real continuous functions of a single independent variable. In § 1 some fundamental notions on functional transformations

* Volterra: "Sopra le funzio. i ch. dipendono da altre funzioni; Sopra le funzioni dipendenti da linee," *Rendiconti della Reale Accademia dei Lincei*, Vol. 3 (1887), five papers. For additional references on the subject of functionals, see the footnotes of our paper.

† Ricci, *Bulletin des Sciences Mathématiques*, Vol. 16 (1892). See also Ricci and Levi-Civita, *Mathematische Annalen*, Vol. 54 (1901); Wright, *Invariants of Quadratic Differential Forms* (Cambridge University Press, 1908); and the Cambridge Tract by O. Veblen.

‡ The literature on these recent researches in differential geometry and differential invariants is extensive. The following books and papers, in addition to the ones already cited, form a fairly complete n-dimensional background for the generalizations of this memoir:

Weyl, *Space—Time—Matter* (English translation of fourth edition), London (1922);

Levi-Civita, *The Absolute Differential Calculus* (English translation), London (1927);

Eisenhart, *Riemmannian Geometry*, Princeton (1926);

Cartan, *La Géométrie des espaces de Riemann*, Paris (1925);

Schouten, *Der Ricci-Kalkül*, Berlin (1924);

Veblen and T. Y. Thomas, "The Geometry of Paths," *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 551-608;

T. Y. Thomas and A. D. Michal, "Differential Invariants of Affinely Connected Manifolds," *Annals of Mathematics*, Vol. 28 (1927), pp. 196-236; "Differential Invariants of Relative Quadratic Differential Forms," *Annals of Mathematics*, Vol. 28 (1927), pp. 631-688.

and functionals are summarized. The connection between differential invariants and algebraic invariants is well known in n-dimensional theories. In the general algebraic invariant theory in n variables, the underlying group of transformations is the linear homogeneous group with non-vanishing determinant. The analogue of this group in function space is the *Fredholm group* of functional transformations with non-vanishing Fredholm determinants. The theory of the Fredholm integral equation is therefore used in § 2 to develop a *functional invariant theory of functional forms.* The discussion is for the most part limited to the quadratic functional form (2. 13). The infinite group of transformations (2. 15) induced in the coefficients of this form by the Fredholm group has a *non-homogeneous* linear character. The theory of functional vectors and functional tensors in function space is given in § 3. It is to be observed that the inversion of the functional tensor law of transformation is effected by solving mixed linear integral equations. Next § 4 is devoted to the generalization of the theory of affine connection and infinitesimal parallelism. The generalized theory of infinitesimal parallelism leads naturally to *functional equations with functional derivatives* and to *integro-differential equations* of static type. The functional curvature tensor and its successive covariant functional derivatives are deduced in § 5 as integrability conditions of *functional equations with functional derivatives.* In connection with this subject the theory of functional operations such as covariant functional differentiation, is developed in such a manner as to make clear the rôle of the functionals that possess differentials of the Volterra form.

The following four paragraphs of this paper are devoted to the development of the function space analogues of various methods and results given by Veblen, Eisenhart, T. Y. Thomas and the author in the cited books and papers. The center of attention is, in the main, the functional invariant theory of the *integro-differential equation* (6. 9). Theorem (7. I) shows the connection between *Fredholm integral equations and normal functional coordinates.* The reader's attention is called to the important remarks made at the end of § 9 in connection with replacement theorems for *affine functional invariants that are not functional tensors.*

A functional invariant theory of quadratic differential functional forms (10. 3) is developed to an extent sufficient for the purposes of an introductory theory of *Riemannian* function space manifolds. *The fundamental functional coefficients* $g_{\alpha\beta}[y^\lambda]$ *of the form* (10. 3) *are not functional tensors.* Minima of functionals, of course, enter into the discussion of § 11.

Because of the extension of tensor notation to integration [see statement immediately following expression (2. 2)] the formulae of this paper attain a form which, it is hoped, is at once both compact and intelligible.

1. *Fundamental notions on function space and functionals.* Let y^x be a real continuous function of a real variable x defined over the interval

$$a \leq x \leq b.$$

The numerical values that y^x assumes while x ranges continuously from a to b may be thought of as the coördinates of a point P in an ideal space which we shall call *function space.* There is of course a certain amount of redundancy in the coördinate system y since the values of y^x at the rational points on the interval (a, b) completely determine the function y^x. This redundancy, however, is of a useful sort as it permits the employment of the remarkable theories of *functions of curves* initiated by Vito Volterra.*

A functional transformation

$$y^x = y^x[\bar{y}^t] \qquad a \leq x \leq b, \tag{1.1}$$

where $y^x[\bar{y}^t]$ is a functional of $\bar{y}^t$ that depends in particular on the variable x and is continuous as a function of x, may be interpreted in two ways. If y^x and $\bar{y}^x$ are the coördinates of two points P and $\bar{P}$ of function space in the *same coördinate system, then* (1, 1) is a point transformation of function space that takes the point $\bar{P}$ into the point P. However, if we interpret y^x and $\bar{y}^x$ to mean the coördinates of the same point P in two different coördinate systems, then (1.1) is a transformation of coördinates that gives the coördinates y^x of a point P in terms of the coördinates $\bar{y}^x$ of P in another coördinate system.

We shall have occasion to use only functional transformations (1.1) in which $y^x[\bar{y}^t]$ is a continuous functional of $\bar{y}^t$ in a region of function space. We shall say that $y^x[\bar{y}^t]$ is a *continuous functional* of $\bar{y}^t$ in a region R of function space if for every $\epsilon > 0$ and every function ϕ^t of R, there exists a positive number δ such that

$$| y^x[\bar{y}^t] - y^x[\phi^t] | < \epsilon \tag{1.2}$$

for all x in (a, b) whenever

$$| \bar{y}^t - \phi^t | < \delta$$

for all t in (a, b), $\bar{y}^t$ being a function that is in R.

Functionals of y^t that depend in particular on several continuous variables will occur frequently throughout this paper. In n-dimensional tensor

* "Sopra le funzioni che dipendono da altre funzioni; Sopra le funzioni dipendenti da linee," *Rendiconti della Reale Accademia dei Lincei*, Vol. 3 (1887). See also his *Leçons sur les équations intégrales* (1913), Chapter I and his *Leçons sur les fonctions de lignes* (1913), Chapters I, II, III, IV.

analysis the convention of letting a subscript stand for a covariant index and a superscript for a contravariant index is widely used. We shall make an analogous distinction between the continuous variables that are written as subscripts and those that appear as superscripts (see § 3). A functional $f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[y^t]$ of y^t that depends in particular on the continuous real variables $\alpha\beta\cdots\gamma$, $\lambda\mu\cdots\nu$ will be said to be a continuous functional of y^t in a region R of our function space if for every $\epsilon > 0$ and every function ϕ^t of R, there exists a positive number δ such that

$$| f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[y^t] - f^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[\phi^t] | < \epsilon$$

for all $\alpha\beta\cdots\gamma$, $\lambda\mu\cdots\nu$ in (a, b) whenever

$$| y^t - \phi^t | < \delta$$

for all t in (a, b), y^t being in R.

Throughout our paper we shall consider functionals which have differentials in the Fréchét sense.* These Fréchét differentials will be assumed in general to be of the Volterra form.

2. *Algebraic functional forms and the Fredholm group.* A special transformation of the form (1.1) is the Fredholm transformation

$$y^i = \bar{y}^i + K_\alpha{}^i \bar{y}^\alpha \tag{2.1}$$

where the quantity $K_\alpha{}^i$ is a continuous real function of the continuous real variables i and α, and the expression $K_\alpha{}^i \bar{y}^\alpha$ stands for

$$\int_a^b K_\alpha{}^i \bar{y}^\alpha \, d\alpha. \tag{2.2}$$

The above convention of letting the repetition of an index in a term, once as a subscript and once as a superscript, stand for a Riemann integration over the interval (a, b) with respect to this index will be used throughout our paper.

Let us assume now that the Fredholm determinant of the kernel $K_\alpha{}^i$ is not equal to zero. Under this assumption we have

$$\bar{y}^i = y^i + k_\alpha{}^i y^\alpha \tag{2.3}$$

where $k_\alpha{}^i$ is the resolvent kernel of $K^{\alpha i}$ which satisfies the well known relations

$$k_\alpha{}^i + K_\alpha{}^i + k_\beta{}^i K_\alpha{}^\beta = 0; \quad k_\alpha{}^i + K_\alpha{}^i + K_\beta{}^i k_\alpha{}^\beta = 0. \tag{2.4}$$

* Cf. Fréchét's paper in the *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 139. See also Evans, *The Cambridge Colloquium*, p. 69 and P. Lévy, *Leçons D'Analyse Fonctionnelle*, pp. 48-105.

The explicit expression for the resolvent kernel $k_a{}^i$ is

$$k_a{}^i = -D_a{}^i[K_\mu{}^\lambda]/D[K_\tau{}^\sigma].$$

In this formula, the functionals $D[K_\mu{}^\lambda]$ and $D_a{}^i[K_\mu{}^\lambda]$ denote the Fredholm determinant of $K_\mu{}^\lambda$ and its first minor respectively. We have the well-known expansions

$$D[K_\mu{}^\lambda] = 1 + K_a{}^a + (1/2!)K^{a_1a_2}_{a_1a_2} + \cdots + (1/p!)K^{a_1a_2\cdots a_p}_{a_1a_2\cdots a_p} + \cdots$$
$$D_j{}^i[K_\mu{}^\lambda] = K_j{}^i + K^{ia_1}_{ja_1} + (1/2!)K^{ia_1a_2}_{ja_1a_2} + \cdots + (1/p!)K^{ia_1a_2\cdots a_p}_{ja_1a_2\cdots a_p} + \cdots$$

in which

$$K^{a_1a_2\cdots a_p}_{\beta_1\beta_2\cdots\beta_p} = \begin{vmatrix} K_{\beta_1}{}^{a_1}, & K_{\beta_2}{}^{a_1}, & \cdots, & K_{\beta_p}{}^{a_1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{\beta_1}{}^{a_p}, & K_{\beta_2}{}^{a_p}, & \cdots, & K_{\beta_p}{}^{a_p} \end{vmatrix}.$$

With these facts in mind one can readily verify that the class of Fredholm transformations (2.1) with non-vanishing Fredholm determinants possesses the group property.*

The differential invariant theories that arise in connection with n-dimensional geometries have a close relation to the classical theory of algebraic forms and algebraic invariants. An analogous situation exists in the functional invariant theories of function space. With this remark in mind we turn now to some fundamental ideas in a theory of functional algebraic forms and functional algebraic invariants.

By a linear functional form we shall understand a linear functional

$$l_a y^a \tag{2.5}$$

which has the law of transformation

$$l_a y^a = \bar{l}_a \bar{y}^a \tag{2.6}$$

under the group of Fredholm transformations (2.1) with non-vanishing determinants. In (2.5), l_a is assumed to be a continuous function of α in the fundamental interval (a, b). As a consequence of (2.1) and (2.6) we obtain

$$\bar{l}_i = l_i + l_\beta K_i{}^\beta. \tag{2.7}$$

Thus the group of Fredholm transformations (2.1) with non-vanishing Fredholm determinants induces in the coefficients l_a an associated group of Fredholm transformations (2.7) with non-vanishing Fredholm determinants.

* I. Fredholm, *Sur une classe d'équations fonctionnelles*, pp. 372-373.

The coefficients $q_{\alpha\beta}$ of a quadratic functional form

$$q_{\alpha\beta}y^\alpha y^\beta; \quad q_{\alpha\beta} = q_{\beta\alpha} \tag{2.8}$$

transform in accordance with the law

$$\bar{q}_{\alpha\beta} = q_{\alpha\beta} + q_{\alpha\gamma} K_\beta^\gamma + q_{\gamma\beta}K_\alpha^\gamma + q_{\gamma\delta}K_\alpha^\gamma K_\beta^\delta. \tag{2.9}$$

This can be proved quite simply by observing that the condition

$$q_{\alpha\beta}y^\alpha y^\beta = \bar{q}_{\alpha\beta}\bar{y}^\alpha\bar{y}^\beta \tag{2.10}$$

leads to a functional identity in $\bar{y}^i$

$$\phi_{\alpha\beta}\bar{y}^\alpha\bar{y}^\beta = 0; \quad \phi_{\alpha\beta} = \phi_{\beta\alpha} \tag{2.11}$$

when we substitute in (2.10) the expression for y^i as given by (2.1). A necessary and sufficient condition that (2.11) be true is that

$$\phi_{\alpha\beta} = 0, \tag{2.12}$$

which is condition (2.9).

The functional form (2.8), however, is not the type of quadratic functional that we wish to consider in detail. Further developments in this paper (particularly with reference to the generalization of n-dimensional Riemannian manifolds) suggest that we consider quadratic functional forms

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\, d\alpha; \quad g_{\alpha\beta} = g_{\beta\alpha}. \tag{2.13}$$

With reference to (2.13) we have the following theorem:

THEOREM 2.I. *A necessary and sufficient condition that*

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\, d\alpha = \bar{g}_{\alpha\beta}\bar{y}^\alpha\bar{y}^\beta + \int_a^b (\bar{y}^\alpha)^2\, d\alpha \tag{2.14}$$

under the group of Fredholm transformations (2.1) *with non-vanishing Fredholm determinants is that*

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\gamma\beta}K_\alpha^\gamma + g_{\alpha\gamma}K_\beta^\gamma + g_{\gamma\delta}K_\alpha^\gamma K_\beta^\delta + K_\alpha^\beta + K_\beta^\alpha + \int_a^b K_\alpha^\gamma K_\beta^\gamma d\gamma. \tag{2.15}$$

In order to obtain $g_{\alpha\beta}$ as a functional of $\bar{g}_{\alpha\beta}$ we have to solve the seemingly complicated integral equation (2.15). The existence, however, of a resolvent kernel k_β^α to K_β^α insures the unicity of the continuous solution

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{g}_{\gamma\beta}k_\alpha^\gamma + \bar{g}_{\alpha\gamma}k_\beta^\gamma + \bar{g}_{\gamma\delta}k_\alpha^\gamma k_\beta^\delta + k_\alpha^\beta + k_\beta^\alpha + \int_a^b k_\alpha^\gamma k_\beta^\gamma d\gamma. \tag{2.16}$$

of (2.15). To establish this, one need only follow a method of proof similar to that used by Volterra * in his treatment of equation (2.1).

DEFINITION OF A RELATIVE SCALAR FUNCTIONAL INVARIANT. *A continuous functional $f[g_{\alpha\beta}]$ of $g_{\alpha\beta}$ will be called a relative scalar functional invariant of weight w of the quadratic functional form* (2.13) *if its law of transformation is*

$$f[\bar{g}_{\alpha\beta}] = (D[K_\nu{}^\mu])^w f[g_{\alpha\beta}] \qquad (w \text{ a constant number}) \tag{2.17}$$

under the group of transformations (2.1).

In (2.17) the quantity $D[K_\nu{}^\mu]$ stands for the Fredholm determinant of the function $K_\nu{}^\mu$ which occurs in (2.15).

We shall establish the following theorem.

THEOREM 2.II. *The Fredholm determinant $D[g_{\alpha\beta}]$ is a relative scalar functional invariant of weight* 2; i.e.,

$$D[\bar{g}_{\alpha\beta}] = (D[K_\nu{}^\mu])^2 D[g_{\alpha\beta}]. \tag{2.18}$$

To prove this theorem we define the quantity $M_{\alpha\beta}$ by

$$M_{\alpha\beta} = g_{\alpha\beta} + K_\beta{}^\alpha + g_{\alpha\delta}K_\beta{}^\delta \tag{2.19}$$

In terms of the quantities $M_{\alpha\beta}$ and $K_\alpha{}^\beta$, the relation (2.15) takes the form

$$\bar{g}_{\alpha\beta} = K_\alpha{}^\beta + M_{\alpha\beta} + K_\alpha{}^\gamma M_{\gamma\beta}. \tag{2.20}$$

By the theorem on the multiplication of Fredholm determinants † we obtain the relations

$$D[\bar{g}_{\alpha\beta}] = D[K_\nu{}^\mu]\, D[M_{\alpha\beta}]. \tag{2.21}$$

But by (2.19) and the same theorem on Fredholm determinants we obtain

$$D[M_{\alpha\beta}] = D[g_{\alpha\beta}]\, D[K_\nu{}^\mu]. \tag{2.22}$$

Our theorem follows immediately on observing that

$$D[K_\beta{}^\alpha] = D[K_\alpha{}^\beta]. \tag{2.23}$$

A quadratic functional form (2.13) has an infinite number of functional invariants when we extend the notion of a functional invariant in the following manner.

---

* Cf. M. Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge Tract No. 10, pp. 21-22.

† I. Fredholm, *Sur une classe d'équations fonctionnelles*, pp. 381-383. See also G. C. Evans, *Cambridge Colloquium*, "Functionals and Their Applications," pp. 122-125.

DEFINITION OF A FUNCTIONAL INVARIANT. *A continuous functional* $f^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}[g_{\alpha\beta}]$ *of the coefficients* $g_{\alpha\beta}$ *of the quadratic functional form* (2.13) *that depends in particular on the variables* $\gamma\delta\cdots\epsilon$, $\lambda\mu\cdots\sigma$ *will be called a relative functional invariant of weight* w *if under the group of Fredholm transformations* (2.1) *with non-vanishing Fredholm determinants we have*

$$f^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}[\bar{g}_{\alpha\beta}] = \phi^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}\left[f^{cd\cdots e}_{lm\cdots s}[g_{\alpha\beta}], g_{ij}, K_q{}^p, k_v{}^u\right](D[K_\beta{}^\alpha])^w \tag{2.24}$$

where $\phi^{\gamma\delta\cdots\epsilon}_{\lambda\mu\cdots\sigma}$ *is a fixed functional of its four functional arguments* $f^{cd\cdots e}_{lm\cdots s}$, g_{ij}, $K_q{}^p$ *and* $k_v{}^u$ *that eliminates the variables* $cd\cdots e$, $lm\cdots s$, ij, p, q, u, v, *and* $k_v{}^u$ *is the resolvent kernel of* $K_v{}^u$.

Clearly the Fredholm determinant $D[g_{\alpha\beta}]$ is a relative functional invariant of weight two. A relative functional invariant of weight zero will be spoken of as an absolute functional invariant. An important absolute functional invariant of (2.13) is the resolvent kernel $g^{\alpha\beta}$ of the coefficients $g_{\alpha\beta}$ of (2.13). In fact we have the theorem.

THEOREM 2.III. *If the Fredholm determinant* $D[g_{\alpha\beta}]$ *of a quadratic functional form*

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\, d\alpha \tag{2.25}$$

is not zero, then the resolvent kernel $g^{\alpha\beta}[g_{ab}]$ *of* $g_{\alpha\beta}$

$$g^{\alpha\beta}[g_{ab}] = -D^{\alpha\beta}[g_{ab}]/D[g_{ab}] \tag{2.26}$$

is an absolute functional invariant of the quadratic form (2.25) *while the first Fredholm minor* $D^{\alpha\beta}[g_{ab}]$ *of the Fredholm determinant* $D[g_{ab}]$ *is a relative functional invariant of weight two. In fact we have*

$$\begin{aligned} \text{(a)}\quad g^{\alpha\beta}[\bar{g}_{ab}] &= g^{\alpha\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}]k_\gamma{}^\alpha + g^{\alpha\gamma}[g_{ab}]k_\gamma{}^\beta + g^{\gamma\delta}[g_{ab}]k_\gamma{}^\alpha k_\delta{}^\beta \\ &\quad + k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\gamma{}^\alpha k_\gamma{}^\beta\, d\gamma \end{aligned} \tag{2.27}$$

$$\begin{aligned} \text{(b)}\quad D^{\alpha\beta}[\bar{g}_{ab}] &= (D[K_\mu{}^\lambda])^2\,\{D^{\alpha\beta}[g_{ab}] + D^{\gamma\beta}[g_{ab}]k_\gamma{}^\alpha \\ &\quad + D^{\alpha\gamma}[g_{ab}]k_\gamma{}^\beta + D^{\gamma\delta}[g_{ab}]k_\gamma{}^\alpha k_\delta{}^\beta \\ &\quad - D[g_{\sigma\tau}]\,(k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\gamma{}^\alpha k_\gamma{}^\beta\, d\gamma)\}. \end{aligned}$$

To prove this theorem we first observe that

$$D[\bar{g}_{\alpha\beta}] \neq 0.$$

This is a direct consequence of (2.18) and of our hypothesis about $D[g_{\alpha\beta}]$.

Thus $g^{\alpha\beta}[\bar{g}_{ab}]$ exists. Let $m_{\alpha\beta}$ be the resolvent kernel of $M_{\alpha\beta}$ and $k_\beta{}^\alpha$ that of $K_\beta{}^\alpha$, then it follows that the resolvent kernel * of

$$K_\alpha{}^\beta + M_{\alpha\beta} + K_\alpha{}^\gamma M_{\gamma\beta}$$

is

$$k_\alpha{}^\beta + m_{\alpha\beta} + \int_a^b m_{\alpha\gamma} k_\gamma{}^\beta \, d\gamma.$$

Hence it is a consequence of (2.20) that

$$g^{\alpha\beta}[\bar{g}_{ab}] = k_\alpha{}^\beta + m_{\alpha\beta}[g_{ab}] + \int_a^b m_{\alpha\gamma}[g_{ab}] k_\gamma{}^\beta \, d\gamma. \tag{2.28}$$

By a similar argument we obtain from (2.19) the relation

$$m_{\alpha\beta}[g_{ab}] = g^{\alpha\beta}[g_{ab}] + k_\beta{}^\alpha + k_\delta{}^\alpha \, g^{\delta\beta}[g_{ab}]. \tag{2.29}$$

Finally on eliminating the functional $m_{\alpha\beta}[g_{ab}]$ in (2.28) by means of (2.29) we obtain equation (a) of (2.27). The relation (b) of (2.27) is readily deduced by multiplying each member of equation (a) of (2.27) by the corresponding member of equation (2.18). This completes the proof of our theorem.

One can proceed now with a study of the invariant nature of the Fredholm minors of any order of the Fredholm determinant $D[g_{ab}]$. Let us, however, turn to certain other questions. First let us inquire into the meaning of the condition

$$K_\alpha{}^\beta + K_\beta{}^\alpha + \int_a^b K_\alpha{}^\gamma K_\beta{}^\gamma \, d\gamma = 0. \tag{2.30}$$

One can readily verify the fact that (2.30) is the necessary and sufficient condition that the functional

$$\int_a^b (y^\alpha)^2 \, d\alpha$$

be invariant under a Fredholm transformation (2.1). Furthermore it is evident from (2.30) that $K_\alpha{}^\beta$ is the resolvent kernel of $K_\beta{}^\alpha$ and that the Fredholm determinant

$$D[K_\beta{}^\alpha] = \pm 1. \tag{2.31}$$

The family of Fredholm transformations (2.1) with non-vanishing Fredholm determinants that satisfy condition (2.30) form a sub-group of the group of all Fredholm transformations with non-vanishing Fredholm determinants.

---

* Cf. G. C. Evans, *loc. cit.*, pp. 121-122.

We shall call this sub-group the *orthogonal Fredholm group of transformations.* As consequences of (2. 30) we have the following formulae

$$K_\beta{}^\alpha + K_\alpha{}^\beta + \int_a^b K_\gamma{}^\alpha K_\gamma{}^\beta \, d\gamma = 0$$

$$k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\gamma{}^\alpha k_\gamma{}^\beta \, d\gamma = 0 \tag{2. 32}$$

$$k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\alpha{}^\gamma k_\beta{}^\gamma \, d\gamma = 0.$$

With the aid of these formulae and of (2. 30) itself a multitude of interesting results flow forth. For example, *under the orthogonal Fredholm group and under that alone do we have*

$$\begin{aligned} &D[\bar{g}_{\alpha\beta}] = D[g_{\alpha\beta}] \\ &\bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\gamma\beta}K_\alpha{}^\gamma + g_{\alpha\gamma}K_\beta{}^\gamma + g_{\gamma\delta}K_\alpha{}^\gamma K_\beta{}^\delta \\ &g^{\alpha\beta}[\bar{g}_{ab}] = g^{\alpha\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}]\, k_\gamma{}^\alpha + g^{\alpha\gamma}[g_{ab}]\, k_\gamma{}^\beta + g^{\gamma\delta}[g_{ab}]\, k_\gamma{}^\alpha k_\delta{}^\beta \\ &D^{\alpha\beta}[\bar{g}_{ab}] = D^{\alpha\beta}[g_{ab}] + D^{\gamma\beta}[g_{ab}]\, k_\gamma{}^\alpha + D^{\alpha\gamma}[g_{ab}]\, k_\gamma{}^\beta + D^{\gamma\delta}[g_{ab}]\, k_\gamma{}^\alpha k_\delta{}^\beta. \end{aligned} \tag{2. 33}$$

3. *Tensor algebra in function space.* In order to develop a theory of tensors in function space, we must first specify the underlying group of functional transformations. Consider a functional transformation

$$y^i = y^i[\bar{y}^\alpha] \tag{3. 1}$$

which takes the functional region

$$|\, \bar{y}^\alpha - \bar{y}_0{}^\alpha \,| < M \tag{3. 1a}$$

of a fixed continuous function $\bar{y}_0{}^\alpha$ into the functional region

$$|\, y^i - y_0{}^i \,| < N \tag{3. 1b}$$

of the fixed continuous function $y_0{}^i$ that corresponds to $\bar{y}_0{}^i$. We shall assume that (3. 1) satisfies the following conditions.

(a) *The functional $y^i[\bar{y}^\alpha]$ is single-valued, continuous as a function of i for $a \leq i \leq b$, and continuous as a functional of $\bar{y}^\alpha$ in the functional region* (3. 1a)

(b) *the functional y^i has a Fréchét differential of the form*

$$\delta y^i = \delta \bar{y}^i + {}'y_\alpha{}^i \,[\bar{y}]\, \delta \bar{y}^\alpha \tag{3. 2}$$

where

$$|\, \delta y^i - \Delta y^i \,| < \epsilon \max |\, \delta \bar{y}^i \,|,$$

ϵ being a quantity that approaches zero with $\max |\, \delta \bar{y}^i \,|$

(c) *in* (3.2) *the functional* $'y_\alpha{}^i[\bar{y}]$ *is assumed to be continuous in* i, α, $\bar{y}^\alpha$ *for* $a \leq i$, $\alpha \leq b$ *and* $|\bar{y}^\alpha - \bar{y}_0{}^\alpha| < M$, *and to have a non-vanishing Fredholm determinant* $D['y_\alpha{}^i]$ *for* $|\bar{y}^\alpha - \bar{y}_0{}^\alpha| < M$.

The totality of functional transformations (3.1) which satisfy the above conditions form a group. In fact if

$$\bar{y}^i = \bar{y}^i[\bar{y}^\alpha]$$

is a functional transformation that takes the region

$$|\bar{y}^i - \bar{y}_0{}^i| < L$$

into the region (3.1a) then

$$y^i = \psi^i[\bar{y}] \quad (y^i[\bar{y}[\bar{y}]] \equiv \psi^i[\bar{y}])$$

is a functional transformation that takes the region

$$|\bar{y}^i - \bar{y}_0{}^i| < L$$

into the region (3.1b). Such a transformation obviously satisfies condition (a). Clearly

$$\delta y^i = \delta\bar{y}^i + {}'\psi_\alpha{}^i[\bar{y}]\,\delta\bar{y}^\alpha$$

where

$${}'\psi_\alpha{}^i[\bar{y}] = {}'y_\alpha{}^i[\bar{y}] + {}'\bar{y}_\alpha{}^i[\bar{y}] + {}'y_\beta{}^i[\bar{y}]\,{}'\bar{y}_\alpha{}^\beta[\bar{y}].$$

Hence

$$|\delta y^i - \Delta\psi^i[\bar{y}]| < \epsilon_1 \max|\delta\bar{y}| + |\Delta\bar{y}^i[\bar{y}] + {}'y_\alpha{}^i[\bar{y}]\,\delta\bar{y}^\alpha + {}'y_\beta{}^i[\bar{y}]'\,\bar{y}_\alpha{}^\beta[\bar{y}]\,\delta\bar{y}^\alpha - \Delta\psi^i[\bar{y}]|.$$

Since

$$|\Delta\bar{y}^i[\bar{y}] + {}'y_\alpha{}^i[\bar{y}]\,\Delta\bar{y}^\alpha[\bar{y}] - \Delta\psi^i[\bar{y}]| < \epsilon_3 \max|\delta\bar{y}|$$

we obtain the inequality

$$|\delta y^i - \Delta\psi^i[\bar{y}]| < (\epsilon_1 + \epsilon_2 + \epsilon_3) \max|\delta\bar{y}|$$

in which ϵ_1, ϵ_2 and ϵ_3 are quantities that approach zero with $\max|\delta\bar{y}^i|$. Thus the transformation $y^i = \psi^i[\bar{y}]$ satisfies condition (b). By Fredholm's theorem on the multiplication of Fredholm determinants we have

$$D['\psi_\alpha{}^i] = D['y_\alpha{}^i]\,D['\bar{y}_\mu{}^\lambda]$$

and hence $D['\psi_\alpha{}^i] \neq 0$ in the region $|\bar{y}^i - \bar{y}_0{}^i| < L$. The group property of the totality of transformations (3.1) follows readily with the aid of the above observations.

In order to have a group of transformations with single-valued inverses throughout a functional region, we shall restrict ourselves to transformations (3. 1) that satisfy the following additional condition:

(d) *the functional transformation* (3. 1) *possesses a unique single-valued inverse*

$$\bar{y}^i = \bar{y}^i\,[y^\alpha] \tag{3. 3}$$

that takes the region (3. 1b) *into the region* (3. 1a) *and that satisfies conditions* (a), (b) *and* (c) *in the region* (3. 1b).

The class of all admitted transformations (3. 1) with their inverses forms a group of functional transformations.

The symbol $\mathfrak{G}$ *will be used to denote an arbitrary transformation of this group of functional transformations.*

Clearly we have

$$\delta\bar{y}^i = \delta y^i + {}'\bar{y}_\alpha{}^i\,[y]\,\delta y^\alpha \tag{3. 4}$$

where ${}'\bar{y}_\alpha{}^i$ is the resolvent kernel of ${}'y_\alpha{}^i$.

The functionals ${}'y_\alpha{}^i[\bar{y}^\beta]$ and ${}'\bar{y}_\alpha{}^i[y^\beta]$ will be by definition the *functional derivatives* of the functionals $y^i[\bar{y}^\beta]$ and $\bar{y}^i[y^\beta]$ respectively.

If we eliminate $\delta\bar{y}^i$ in (3. 2) by means of (3. 4) we obtain the identity

$${}'y_\alpha{}^i[\bar{y}]\delta y^\alpha + {}'\bar{y}_\alpha{}^i[y]\delta y^\alpha + {}'y_\beta{}^i[\bar{y}]\,{}'\bar{y}_\alpha{}^\beta[y]\delta y^\alpha = 0.$$

From the arbitrariness of δy^α we have the fundamental relation

$${}'y_\alpha{}^i[\bar{y}] + {}'\bar{y}_\alpha{}^i[y] + {}'y_\beta{}^i[\bar{y}]\,{}'\bar{y}_\alpha{}^\beta[y] = 0. \tag{3. 5}$$

By a similar argument we obtain the second fundamental relation

$${}'\bar{y}_\alpha{}^i[y] + {}'y_\alpha{}^i[\bar{y}] + {}'\bar{y}_\beta{}^i[y]\,{}'y_\alpha{}^\beta[\bar{y}] = 0. \tag{3. 6}$$

on first eliminating δy^i in (3. 4) by means of (3. 2). Clearly the identities (3. 5) and (3. 6) are the well known relations that hold between a kernel of an integral equation and its resolvent kernel.

Let $f[y^\alpha]$ be a continuous functional of y^α that is a *scalar;* i. e.,

$$f[y^\alpha] = \bar{f}[\bar{y}^\alpha] \tag{3. 7}$$

under a transformation $\mathfrak{G}$. Furthermore let it possess a differential of the Volterra form

$$\delta f[y] = {}'f_\alpha[y]\,\delta y^\alpha \tag{3. 8}$$

where $'f_\alpha$ is continuous both as a functional of y^α and as a function of α. By definition, $'f_\alpha[y]$ shall also be called the *functional derivative* of the functional $f[y]$ with respect to the argument y^β.

From (3.7) we obtain

$$'\bar{f}_\alpha[\bar{y}]\,\delta\bar{y}^\alpha = 'f_\beta[y]\,'y_\alpha{}^\beta[\bar{y}]\,\delta\bar{y}^\alpha + 'f_\alpha[y]\,\delta\bar{y}^\alpha.$$

Hence the functional derivative $'\bar{f}_\alpha[\bar{y}]$ in the variable $\bar{y}^\gamma$ is given in terms of the functional derivative $'f_\alpha[y]$ in the variable y^γ by the formula

$$'\bar{f}_\alpha[\bar{y}] = 'f_\alpha[y] + 'f_\beta[y]\,'y_\alpha{}^\beta[\bar{y}]. \tag{3.9}$$

The functional derivative $'f_\alpha[y]$ of a scalar functional $f[y]$ is therefore a covariant functional vector. This result is in accordance with the following definition.

DEFINITION OF A COVARIANT FUNCTIONAL VECTOR:* *A functional $\xi_i[y^\alpha]$ will be said to be a covariant functional vector if its law of transformation under the group of transformations* $\mathfrak{G}$ *is given by*

$$\bar{\xi}_\alpha[\bar{y}] = \xi_\alpha[y] + \xi_\beta[y]\,'y_\alpha{}^\beta[\bar{y}]. \tag{3.10}$$

The formula (3.10) has the three important properties:

(a) $\bar{\xi}_\alpha$ is a linear homogeneous functional of ξ_α

(b) If ξ_α vanishes in one coördinate system y^i it vanishes in every coördinate system

(c) the group property.

To prove property (c), and incidentally derive an interesting formula, let us consider the transformations

$$\begin{aligned} \bar{\bar{\xi}}_\alpha[\bar{\bar{y}}] &= \bar{\xi}_\alpha[\bar{y}] + \bar{\xi}_\beta[\bar{y}]\,'\bar{y}_\alpha{}^\beta[\bar{\bar{y}}] \\ \bar{\xi}_\alpha[\bar{y}] &= \xi_\alpha[y] + \xi_\gamma[y]\,'y_\alpha{}^\gamma[\bar{y}]. \end{aligned} \tag{3.11}$$

From (3.11) we obviously obtain the relation

$$\bar{\bar{\xi}}_\alpha[\bar{\bar{y}}] = \xi_\alpha[y] + \xi_\beta[y]\,\left('y_\alpha{}^\beta[\bar{y}] + '\bar{y}_\alpha{}^\beta[\bar{\bar{y}}] + 'y_\gamma{}^\beta[\bar{y}]\,'\bar{y}_\alpha{}^\gamma[\bar{\bar{y}}]\right). \tag{3.12}$$

But

$$'z_\alpha{}^\beta[\bar{\bar{y}}] = 'y_\alpha{}^\beta[\bar{y}] + '\bar{y}_\alpha{}^\beta[\bar{\bar{y}}] + 'y_\gamma{}^\beta[\bar{y}]\,'\bar{y}_\alpha{}^\gamma[\bar{\bar{y}}],\quad z^\beta[\bar{\bar{y}}^\alpha] = y^\beta[\bar{y}^\gamma[\bar{\bar{y}}^\alpha]] \tag{3.13}$$

* If we use the geometrical terminology we may think of (3.10) as a formula which gives the infinitely many components of the functional vector ξ_α at a point P in the coördinate system $\bar{y}^i$ in terms of its components at the same point P but in a different coördinate system y^i.

as is shown most easily by observing that ${}'z_\alpha{}^\beta[\bar{y}]$ as given by (3. 13) is the kernel of the product of two Fredholm transformations of the type (3. 2). Furthermore, the unique continuous solution of the Fredholm integral equation (3. 10) is

$$\xi_\alpha[y] = \bar{\xi}_\alpha[\bar{y}] + \bar{\xi}_\beta[\bar{y}]\, {}'\bar{y}_\alpha{}^\beta[y] \tag{3. 14}$$

since ${}'\bar{y}_\alpha{}^\beta[y]$ is the resolvent kernel of ${}'y_\alpha{}^\beta[\bar{y}]$. Finally the identical transformation

$$y^i = \bar{y}^i$$

implies

$${}'y_\alpha{}^i = 0; \quad {}'\bar{y}_\alpha{}^i = 0.$$

But these conditions reduce (3. 10) and (3. 14) to the forms

$$\bar{\xi}_\alpha[\bar{y}] = \xi_\alpha[y], \quad \xi_\alpha[y] = \bar{\xi}_\alpha[\bar{y}]$$

respectively. The proof of (c) is therefore complete.

Formula (3. 13) shows that the functional derivative of a transformation $\mathfrak{G}$ does not transform like a covariant functional vector. In fact let us consider any scalar functional $s_i[y^\alpha]$ that depends in particular on a variable i and has a differential of the form

$$\delta s_i = \delta y^i + {}'s_{i,\alpha}[y]\, \delta\bar{y}^\alpha \tag{3. 15}$$

where ${}'s_{i,\alpha}[y]$ denotes the functional derivative of s_i. Throughout our paper we shall understand that *subscripts following a comma denote variables that arise by the process of functional differentiation.* By a reasoning similar to the one used in obtaining (3. 9) we find that

$${}'\bar{s}_{i,\alpha}[\bar{y}] = {}'s_{i,\alpha}[y] + {}'s_{i,\beta}\, {}'y_\alpha{}^\beta[\bar{y}] + {}'y_\alpha{}^i[\bar{y}]. \tag{3. 16}$$

The law of transformation (3. 16) possesses the group property. However, if the scalar functional has a differential of the Volterra form

$$\delta s_i = {}'s_{i,\alpha}[y]\, \delta y^\alpha$$

${}'s_{i,\alpha}$ will be a covariant functional vector for each value of i.

DEFINITION OF A CONTRAVARIANT FUNCTIONAL VECTOR: *A functional $\xi^i[y^\alpha]$ will be said to be a contravariant functional vector if its law of transformation under the group of transformations $\mathfrak{G}$ is given by*

$$\bar{\xi}^\alpha[\bar{y}] = \xi^\alpha[y] + \xi^\beta[y]\, {}'\bar{y}_\beta{}^\alpha[y]. \tag{3. 17}$$

The unique continuous solution of the Fredholm integral equation (3. 17) is

$$\xi^\alpha[y] = \bar{\xi}^\alpha[\bar{y}] + \bar{\xi}^\beta[\bar{y}]\, {}'y_\beta{}^\alpha[\bar{y}]. \tag{3. 18}$$

It is evident from (3.2) and (3.4) that the differential δy^i of a transformation $\mathfrak{G}$ transforms according to the contravariant vector law.

If $\xi_i[y]$ and $\eta^i[y]$ are covariant and contravariant functional vectors then

$$\xi_i \eta^i$$

is a scalar functional, i. e.,

$$\bar{\xi}_i[\bar{y}]\,\bar{\eta}^i[\bar{y}] = \xi_i[y]\,\eta^i[y]. \tag{3.19}$$

This can be shown by a direct calculation by means of the defining relations (3.10) and (3.17), and the functional identity (3.5). Conversely if (3.19) holds for an arbitrary contravariant functional vector η^i then ξ_i is of necessity a covariant functional vector. Also if (3.19) holds for an arbitrary covariant functional vector ξ_i then η^i is necessarily a contravariant functional vector.

The above methods suggest a way of obtaining the law of transformation of more complicated functional quantities. Suppose that

$$T_\beta{}^\alpha[\bar{y}]\,\bar{\xi}_\alpha[\bar{y}]\,\bar{\eta}^\beta[\bar{y}] = T_\beta{}^\alpha[y]\,\xi_\alpha[y]\,\eta^\beta[y] \tag{3.20}$$

under the group of transformations $\mathfrak{G}$ for arbitrary covariant functional vectors ξ_α and for arbitrary contravariant functional vectors η^α. If in (3.20) we substitute for $\xi_\alpha[y]$ and $\eta^\beta[y]$ their expressions in terms of $\bar{\xi}_\alpha[\bar{y}]$ and $\bar{\eta}^\beta[\bar{y}]$ respectively we obtain a functional identity of the form

$$\phi_\beta{}^\alpha\,\bar{\xi}_\alpha[\bar{y}]\,\bar{\eta}^\beta[\bar{y}] = 0 \tag{3.21}$$

where $\phi_\beta{}^\alpha$ is independent of $\bar{\xi}_\alpha$ and $\bar{\eta}^\beta$. Hence by a double use of the fundamental lemma of the calculus of variations we obtain the condition

$$\phi_\beta{}^\alpha = 0. \tag{3.22}$$

The explicit form of this condition is

$$\overline{T}_\beta{}^\alpha[\bar{y}] = T_\gamma{}^\alpha[y]\,{}'y_\beta{}^\gamma[\bar{y}] + T_\beta{}^\alpha[y] + {}'\bar{y}_\delta{}^\alpha[y]\,(T_\gamma{}^\delta[y]\,{}'y_\beta{}^\gamma[\bar{y}] + T_\beta{}^\delta[y]). \tag{3.23}$$

On multiplying both sides of (3.23) by ${}'y_\alpha{}^i[\bar{y}]$ and integrating with respect to α, we obtain an equivalent but more compact form of (3.23)

$$\overline{T}_\beta{}^\alpha[\bar{y}]\,{}'y_\alpha{}^i = -{}'\bar{y}_\delta{}^i\,(T_\gamma{}^\delta[y]\,{}'y_\beta{}^\gamma + T_\beta{}^\delta[y]). \tag{3.24}$$

This result was obtained by making use of the fundamental functional identity (3.5).

A functional $T_\beta{}^\alpha[y]$ whose law of transformation under the group of transformations $\mathfrak{G}$ is (3.24) will be called a *functional tensor* (or merely

tensor) of order two. This particular tensor T_β^α is covariant of order one and contravariant of order one.

GENERAL DEFINITION OF A FUNCTIONAL TENSOR: *A functional* $T^{ijkl\cdots q}_{abc\cdots g}[y^\sigma]$ *that depends in particular on the variables* $i, j, \cdots, q$; $a, b, \cdots, g$ *will be by definition a functional tensor if its law of transformation under the group of transformations* $\mathfrak{G}$ *is given by*

$$
\begin{aligned}
T^{ijkl\cdots pq}_{abc\cdots g}[\bar{y}] &= \left(^{ijkl\cdots pq}_{abc\cdots g}\right) + {}'\bar{y}_a{}^i[y]\left(^{ajkl\cdots pq}_{abc\cdots g}\right) + {}'y_a{}^j\left(^{iakl\cdots pq}_{abc\cdots g}\right) \\
&+\cdots+ {}'\bar{y}_a{}^q\left(^{ijkl\cdots pa}_{abc\cdots g}\right) + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^j\left(^{a\beta kl\cdots pq}_{abc\cdots g}\right) + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^k\left(^{aj\beta l\cdots pq}_{abc\cdots g}\right) \\
&+\cdots+ {}'\bar{y}_a{}^p\,{}'\bar{y}_\beta{}^q\left(^{ijkl\cdots a\beta}_{abc\cdots g}\right) +\cdots+\cdots \\
&\qquad + {}'\bar{y}_a{}^i\,{}'\bar{y}_\beta{}^j\,{}'\bar{y}_\gamma{}^k\cdots{}'\bar{y}_\pi{}^p\,{}'\bar{y}_\rho{}^q\left(^{a\beta\gamma\cdots\pi\rho}_{abc\cdots g}\right)
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\left(^{ijkl\cdots pq}_{abc\cdots fg}\right) &= T^{ij\cdots q}_{ab\cdots g}[y] + {}'y_a{}^a[\bar{y}]\,T^{ij\cdots q}_{ab\cdots g} + {}'y_b{}^a\,T^{ij\cdots q}_{aa\cdots g} \\
&\qquad +\cdots+ {}'y_g{}^a\,T^{ij\cdots q}_{ab\cdots a} \\
&+ {}'y_a{}^a\,{}'y_b{}^\beta\,T^{ij\cdots q}_{a\beta c\cdots g} + {}'y_a{}^a\,{}'y_c{}^\beta\,T^{ij\cdots q}_{ab\beta\cdots g} +\cdots+ {}'y_f{}^a\,{}'y_g{}^\beta\,T^{ij\cdots q}_{abc\cdots a\beta} \\
&+\cdots+\cdots + {}'y_a{}^a\,{}'y_b{}^\beta\,{}'y_c{}^\gamma\cdots{}'y_f{}^\zeta\,{}'y_g{}^\eta\,T^{ij\cdots q}_{ab\gamma\cdots\zeta\eta}\,.
\end{aligned}
\tag{3.26}
$$

The functional tensor T will be referred to as one of order $\kappa+\nu$, covariant of order κ and contravariant of order ν when κ is the number of variables $abc\cdots fg$ and ν the number of variables $ijk\cdots pq$. Thus there are $2^{\kappa+\nu}$ terms on the right side of (3.25).

The law of transformation of a functional tensor enjoys the properties (a), (b) and (c) of page 485. Furthermore, symmetry or skew-symmetry relations in the continuous variables $ij\cdots q$, $ab\cdots g$ are not altered by a change of the variable y^i.

Let $H_a[y]$ be a covariant functional vector that possesses a functional derivative ${}'H_{a,\beta}[y]$. Then as Volterra * has shown, the integrability conditions of the functional equation

$$\delta f[y] = H_a[y]\,\delta y^a, \tag{3.27}$$

with the functional $f[y]$ as the unknown, are

$${}'H_{a,\beta} - {}'H_{\beta,a} = 0. \tag{3.28}$$

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* Cf. his *Leçons sur les fonctions de lignes*, Paris (1913), pp. 27-28.

It is a consequence of the law of transformation of the vector H_a that $'H_{a,\beta} - 'H_{\beta,a}$ is a covariant functional tensor of order two. Hence (3.28) is a tensor functional equation; i. e., if it holds in a variable y^i, it continues to hold in any other variable $\bar{y}^i$.

From the functional tensor $\phi_{\alpha\beta}[y]$ where

$$\phi_{\alpha\beta} = 'H_{\alpha,\beta} - 'H_{\beta,\alpha}$$

we can form the covariant functional tensor $\phi_{\alpha\beta\gamma}[y]$ of order three where

$$\phi_{\alpha\beta\gamma}[y] = '\phi_{\alpha\beta,\gamma}[y] - '\phi_{\gamma\beta,\alpha}[y] - '\phi_{\alpha\gamma,\beta}[y].$$

Obviously

$$\phi_{\alpha\beta\gamma}[y] \equiv 0.$$

More generally, let $\psi_{\alpha\beta\gamma\ldots\kappa\lambda}[\dot{y}]$ be an alternating covariant functional tensor, i. e.,

$$\psi_{fgh\ldots mn} = \psi_{\alpha\beta\gamma\ldots\kappa\lambda} \quad \text{or} \quad -\psi_{\alpha\beta\gamma\ldots\kappa\lambda}$$

according as $fgh\cdots mn$ is an even or an odd permutation of $\alpha\beta\gamma\cdots\kappa\lambda$. The functional $\psi_{\alpha\beta\gamma\ldots\kappa\lambda}[y]$ defined by

$$(3.29)\qquad \psi_{\alpha\beta\gamma\ldots\kappa\lambda\mu}[y] = '\psi_{\alpha\ldots\lambda,\mu}[y] - \psi_{\mu\beta\gamma\ldots\kappa\lambda,\alpha}[y] - '\psi_{\alpha\mu\gamma\ldots\kappa\lambda,\beta}[y] - \cdots - '\psi_{\alpha\beta\gamma\ldots\kappa\mu,\lambda}[y]$$

is an alternating covariant functional tensor of one higher order than $\psi_{\alpha\beta\gamma\ldots\kappa\lambda}[y]$. If now we form the functional tensor $\psi_{\alpha\beta\gamma\ldots\kappa\lambda\mu\nu}[y]$ from $\psi_{\alpha\beta\gamma\ldots\kappa\lambda\mu}[y]$ according to the rule (3.29) we shall find that *

$$\psi_{\alpha\beta\gamma\ldots\kappa\lambda\mu\nu}[y] \equiv 0.$$

4. *Affinely connected function space manifolds.* Levi-Cività's † notion of infinitesimal parallelism in n-dimensional Riemannian geometry and its subsequent extension by Weyl ‡ to more general affinely connected n-dimensional manifolds has played a significant part in the recent developments of

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* These results suggest the development of a theory of invariant functions of r-dimensional manifolds in *function space* under continuous groups of *functional* transformations. I have already developed a theory of invariant functions of r-dimensional manifolds in n-dimensional space under continuous groups of n-dimensional point transformations. See my paper in *The Transactions of the American Mathematical Society*, July (1927), pp. 612-646.

† Nozione di parallelismo in una varietà qualunque, *Rendiconti Circolo Mathematico Palermo*, Vol. 42 (1927), pp. 173-205.

‡ *Raum, Zeit, Materie* (4th edition, p. 100).

new differential geometries. In this section of our paper we shall develop a theory of affine connection and infinitesimal parallelism in our function space.

Let $y^i(s)$ stand for a function of the two variables i and s where s is to be considered as a parameter that plays a different rôle from the variable i. The function $y^i(s)$ is assumed to be a continuous function of i and an *analytic function* of the parameter s for $a \leq i \leq b$ and $s_0 \leq s \leq s_1$. We may think then of such a function $y^i(s)$ as giving rise to a one dimensional variety or curve in our function space. For a fixed value of s we have a continuous function of i which gives the coördinates of a point P_s on our curve. As s ranges over the interval (s_0, s_1) the point P_s traces the curve in question, where, of course, $y^i(s_0)$ and $y^i(s_1)$ are the coördinates of the initial and final points respectively of our curve. An equation

$$y^i = y^i(s) \tag{4.1}$$

is then the equation of an analytic *parameterized curve* in function space. The same curve will be represented by

$$y^i = y^i(s(t)) \tag{4.2}$$

when $s = s(t)$ is the analytic inverse of an arbitrary analytic substitution $t = t(s)$.

More generally a parameterized r-dimensional variety in our function space will be determined by the equation

$$y^i = y^i\,(u_1, u_2, \cdots, u_r) \tag{4.3}$$

where $y^i(u_1, u_2, \cdots, u_r)$ is continuous as a function of i and analytic as a function of the parameters $u_1, u_2, \cdots, u_r$. The r-dimensional variety (the real geometrical object) may have *any one* of the infinitude of equations

$$y^i = y^i(u_1(v_1, v_2, \cdots, v_r), \cdots, u_r(v_1, \cdots, v_r)) \tag{4.4}$$

as its defining equations. In (4.4), the functions $u_1(v_1, \cdots, v_r), \cdots, u_r(v_1, \cdots, v_r)$ are r arbitrary analytic functions of $v_1, \cdots, v_r$ that are functionally independent.

Consider a functional $H^i{}_{\alpha\beta}[y^k]$ of y^k that depends in particular on the variables i, α, β and has differentials

$$\delta^r H^i{}_{\alpha\beta}[y^k] \qquad (r = 1, 2, 3, \cdots) \tag{4.5}$$

of the Volterra form

$${}^{(r)}H^i{}_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}[y^k]\ \delta y^{\gamma_1}\delta y^{\gamma_2}\cdots\delta y^{\gamma_r}. \tag{4.6}$$

In (4.6), the functional ${}^{(r)}H^i_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}[y^k]$ is assumed to be continuous as a functional of y^k and as a function of $i\alpha\beta\gamma_1\gamma_2\cdots\gamma_r$, and, without loss of generality, symmetric in the variables $\gamma_1\gamma_2\cdots\gamma_r$. We shall speak of ${}^{(r)}H^i_{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_r}$ as the rth functional derivative of $H^i{}_{\alpha\beta}$.

We shall say that a contravariant functional vector $\xi^i[y]$ undergoes a *general linear displacement* from the point with coördinates y^i to a point with coördinates $y^i + \delta y^i$ if the vector ξ^i at the point y^i becomes the vector $\xi^i + \Delta\xi^i$ at the point $y^i + \delta y^i$ where

$$\Delta\xi^i = \delta\xi^i[y] + H^i{}_{\alpha\beta}\,\xi^\alpha\,\delta y^\beta \tag{4.7}$$

as far as terms of the first order in δy^α. In (4.7), $\delta\xi^i$ is the differential of the functional ξ^i

$$\delta\xi^i[y] = {}'\xi_\alpha{}^i[y]\,\delta y^\alpha. \tag{4.8}$$

A functional $H^i{}_{\alpha\beta}[y]$ will be called an *affine connection.* We have thus endowed our amorphous function space with certain geometrical properties by the introduction of an affine connection $H^i{}_{\alpha\beta}[y]$. A function space possessing an affine connection will be called an *affinely connected function space manifold.*

We shall see presently that the vector law of transformation of $\Delta\xi^i[y]$ determines the law of transformation of an affine connection $H^i{}_{\alpha\beta}[y]$. In order to derive this law of transformation, it is necessary to employ a more restricted underlying group of functional transformations. We shall deal with a group of functional transformations in which each transformation satisfies conditions (a), (b), (c) and (d) of page — as well as the following additional requirement:

(e) *in the functional neighborhood* $|\bar{y}^i - \bar{y}_0{}^i| < M$, *the functional derivative* ${}'y_\alpha{}^i[\bar{y}]$ *possesses Fréchet differentials*

$$\delta^p\,{}'y_\alpha{}^i[\bar{y}] \qquad (p = 1, 2, \cdots, s) \tag{4.9}$$

of the Volterra form

$${}^{(p+1)}y^i_{\alpha\beta_1\cdots\beta_p}[\bar{y}]\;\delta\bar{y}^{\beta_1}\cdots\delta\bar{y}^{\beta_p}, \tag{4.10}$$

where the functional ${}^{(p+1)}y^i_{\alpha\beta_1\cdots\beta_p}[\bar{y}]$, *the* $(p+1)$*st functional derivative of* $y^i[\bar{y}]$, *is a symmetric function of the variables* $\alpha\beta_1\cdots\beta_p$, *and is continuous as a functional of* $\bar{y}^\gamma$ *and as a function of* $i\alpha\beta\cdots\beta_p$.

If s is the highest order of the differentials in (4.9), the symbol $\mathfrak{G}_s$ will

be used to denote an arbitrary functional transformation that belongs to such a group.

Under the group of transformations $\mathfrak{G}_1$ we obviously have for a contravariant vector $\xi^i[y]$

$$\begin{aligned}(4.11)\qquad &\xi^i[y] = \bar{\xi}^i[\bar{y}] + {}'y_a{}^i\,[\bar{y}]\;\bar{\xi}^a[\bar{y}]\\ &\delta\,\xi^i[y] = \delta\,\bar{\xi}^i + {}'y_a{}^i\,\delta\bar{\xi}^a + {}''y^i{}_{jk}[\bar{y}]\;\bar{\xi}^j\,\delta\bar{y}^k.\end{aligned}$$

Hence from (4.7) we have

$$\begin{aligned}(4.12)\qquad \Delta\xi^i = \delta\bar{\xi}^i + {}'y_a{}^i\,\delta\bar{\xi}^a + ({}''y^i{}_{jk} + H^i{}_{jk} + H^i{}_{ak}\,{}'y_j{}^a\\ + H^i{}_{ja}\,{}'y_k{}^a + H^i{}_{a\beta}\,{}'y_j{}^a\,{}'y_k{}^\beta)\;\bar{\xi}^j\,\delta\bar{y}^k.\end{aligned}$$

A necessary and sufficient condition that (4.12) be equivalent to

$$\Delta\xi^i = \Delta\bar{\xi}^i + {}'y_a{}^i\,\Delta\bar{\xi}^a,$$

where

$$\Delta\bar{\xi}^i = \delta\bar{\xi}^i[y] + \bar{H}_{a\beta}{}^i[\bar{y}]\;\bar{\xi}^a\,\delta\bar{y}^\beta,$$

for arbitrary vectors ξ^i and differentials $\delta\bar{y}^k$ is that

$$\begin{aligned}(4.13)\qquad H^i{}_{jk}[\bar{y}] + H^a{}_{jk}\,{}'y_a{}^i = {}''y^i{}_{jk} + H^i{}_{jk}[y] + H^i{}_{ak}\,{}'y_j{}^a\\ + H^i{}_{ja}\,{}'y_k{}^a + H^i{}_{a\beta}\,{}'y_j{}^a\,{}'y_k{}^\beta.\end{aligned}$$

Multiplying both sides of (4.13) by ${}'\bar{y}_i{}^\beta[y]$ and integrating with respect to the variable i we obtain the relation

$$\begin{aligned}(4.14)\qquad \bar{H}^\gamma{}_{jk}\,{}'y_\gamma{}^i = -\,{}'\bar{y}_\gamma{}^i\;({}''y^\gamma{}_{jk} + H^\gamma{}_{jk} + H^\gamma{}_{ak}\,{}'y_j{}^a + H^\gamma{}_{ja}\,{}'y_k{}^a\\ + H^\gamma{}_{a\beta}\,{}'y_j{}^a\,{}'y_k{}^\beta).\end{aligned}$$

Another important equivalent form of (4.13) can be obtained by solving the Fredholm integral equation (4.13) with kernel ${}'y_a{}^i$. The unique continuous solution (since the Fredholm determinant $D[{}'y_a{}^i] \neq 0$) is

$$\begin{aligned}(4.15)\qquad \bar{H}_{jk}{}^i = {}''y_{jk}{}^i + H_{jk}{}^i + H^i{}_{ak}\,{}'y_j{}^a + H^i{}_{ja}\,{}'y_k{}^a + H^i{}_{a\beta}\,{}'y_j{}^a\,{}'y_k{}^\beta\\ + {}'\bar{y}_\gamma{}^i\;({}''y^\gamma{}_{jk} + H^\gamma{}_{jk} + H^\gamma{}_{ak}\,{}'y_j{}^a + H^\gamma{}_{ja}\,{}'y_k{}^a + H^\gamma{}_{a\beta}\,{}'y_j{}^a\,{}'y_k{}^\beta).\end{aligned}$$

The law of transformation of an affine connection $H^i{}_{jk}[y]$ can thus be given in any one of the three equivalent forms (4.13), (4.14), (4.15). It is clear from (4.15) that an affine connection $H^i{}_{jk}$ transforms like a functional tensor when and only when

$$(4.16)\qquad {}''y^\gamma{}_{jk}[\bar{y}] = 0.$$

For example, a linear functional transformation

$$y^i = \bar{y}^i + a^i + \beta^i{}_k\bar{y}^k$$

satisfies condition (4.16).

Any affine connection $H^i{}_{jk}$ can be written as

$$H^i{}_{jk} = \Gamma^i{}_{jk}[y] + \Omega^i{}_{jk}[y] \tag{4.17}$$

where

$$\Gamma^i{}_{jk}[y] = \Gamma^i{}_{kj}[y] \tag{4.18}$$

and

$$\Omega^i{}_{jk}[y] = -\Omega^i{}_{kj}[y]. \tag{4.19}$$

The functional *tensor* $\Omega^i{}_{jk}$ will be called the *torsion* of our affinely connected manifold. From now on in this paper, we shall deal with affinely connected function space manifolds whose torsion tensor $\Omega^i{}_{jk}$ is identically zero. This then means that our affine connection $\Gamma^i{}_{jk}[y]$ is a symmetric function of the variables j and k.

We shall say that a functional contravariant vector $\xi[y]$ undergoes a *parallel displacement* from the point with coördinates y^i to a neighboring point with coördinates $y^i + \delta y^i$ if $\Delta\xi^i = 0$, i. e., if

$$\delta\xi^i[y] = -\Gamma^i{}_{\alpha\beta}[y]\xi^\alpha\delta y^\beta, \tag{4.20}$$

where $\Gamma^i{}_{\alpha\beta}[y]$ is the symmetric affine connection of our affinely connected function space manifold.

The functional total differential equation (4.20) is equivalent to the functional equation with functional derivatives

$$'\xi_\beta{}^i[y] = -\Gamma^i{}_{\alpha\beta}[y]\xi^\alpha[y]. \tag{4.21}$$

The integrability conditions * of equation (4.21) are deduced by imposing the condition that the differential $\delta\,'\xi_\beta{}^i[y]$ be a self-adjoint linear functional * of δy^i. A necessary and sufficient condition that $\delta'\xi_\beta{}^i[y]$ be a self-adjoint functional of δy^i is that

$$\xi^\alpha('\Gamma^i{}_{\alpha\beta,\gamma}\ [y] - '\Gamma^i{}_{\alpha\gamma,\beta} + \Gamma^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{\alpha\gamma}\Gamma^i{}_{\sigma\beta})z^\alpha w^\beta = 0 \tag{4.22}$$

for all continuous functions z^α and w^β of sufficiently small moduli. Hence a necessary and sufficient condition that (4.22) hold is that

$$\xi^\alpha[y]B^i{}_{\alpha\beta\gamma}[y] = 0 \tag{4.23}$$

where

$$B^i{}_{\alpha\beta\gamma}[y] = '\Gamma^i{}_{\alpha\beta,\gamma}[y] - '\Gamma_{\alpha\gamma,\beta} + \Gamma^\sigma{}_{\alpha\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{\alpha\gamma}\Gamma^i{}_{\sigma\beta}. \tag{4.24}$$

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* For the theory of the integrability conditions of functional equations and for the concept of a self-adjoint functional see P. Lévy, *loc. cit.*, pp. 91-94, 151-170 and and G. C. Evans, *loc. cit.*, pp. 17-25.

The functional $B^i{}_{\alpha\beta\gamma}$ plays an important rôle in the theory of affinely connected function space manifolds. We shall see in the next paragraph that $B^i{}_{\alpha\beta\gamma}$ is a functional tensor. We may use the terminology of n-dimensional geometry and call the functional $B^i{}_{\alpha\beta\gamma}[y]$ the *curvature tensor* of the affinely connected manifold determined by the affine connection $\Gamma^i{}_{\alpha\beta}[y]$.

Let

$$y^i = y^i(s)$$

be the equation of an arbitrarily given parameterized curve with $s = s_0$ and $s = s_1$, as the values of the parameters corresponding to the initial and final points of the curve. It is clear from (4. 20) that for a parallel displacement of the vector $\xi^i[y]$ along the given curve we must have *

$$\frac{\partial\, \xi^i[y^\alpha(s)]}{\partial s} = -\,\Gamma^i{}_{\alpha\beta}[y^\gamma(s)]\; \xi^\alpha[y^\lambda(s)]\; \frac{\partial\, y^\beta(s)}{\partial s}\ . \tag{4. 25}$$

Since the curve is fixed we see that (4. 25) is a linear *integro-differential equation* of static type

$$\frac{\partial \phi^i(s)}{\partial s} = \psi_\alpha{}^i(s)\; \phi^\alpha(s) \tag{4. 26}$$

with $\phi^i(s)$ as the unknown function of the two variables i and s, and $\psi^i{}_\alpha(s)$ as the known function of the three variables i, α and s. Let

$$\xi^i[y^\alpha(s)] = \phi^i(s)$$

for the given curve. Under sufficiently restrictive conditions, the unique solution $\phi^i(s)$, continuous in i and analytic in s, of the integro-differential equation (4. 25) which takes on the initial value

$$\eta^i = \phi^i(s_0) \tag{4. 27}$$

will be

$$\phi^i(s) = \eta^i + \eta^\omega \Big[(s - s_0)\, {}_{(1)}\lambda^i{}_\omega(s_0) + \frac{(s-s_0)^2}{2!}\, {}_{(2)}\lambda^i{}_\omega(s_0) + \frac{(s-s_0)^3}{3!}\, {}_{(3)}\lambda^i{}_\omega(s_0) + \cdots \Big] \tag{4. 28}$$

where

$${}_{(1)}\lambda^i{}_\omega(s) = -\,\Gamma^i{}_{j\omega}\, \frac{\partial y^j(s)}{\partial s} \tag{4. 29}$$

$${}_{(2)}\lambda^i{}_\omega(s) = \Gamma^i{}_{j\alpha}\, \Gamma^\alpha{}_{k\omega}\, \frac{\partial y^j}{\partial s}\, \frac{\partial y^k}{\partial s} - {}'\Gamma^i{}_{j\omega,k}\, \frac{\partial y^j}{\partial s}\, \frac{\partial y^k}{\partial s} - \Gamma^i{}_{j\omega}\, \frac{\partial^2 y^j}{\partial s^2}$$

and in general

$${}_{(m)}\lambda^i{}_\omega(s) = \frac{\partial\, {}_{(m-1)}\lambda^i{}_\omega(s)}{\partial s} - {}_{(m-1)}\lambda^i{}_\alpha(s)\; \Gamma^\alpha{}_{j\omega}\, \frac{\partial y^j(s)}{\partial s}\ . \tag{4. 30}$$

* Cf. § 6 with particular reference to statements about formula (6, 3).

The solution (4.28) will depend in general on the choice of the function $y^i(s)$ although it will be independent of the parameterization s. This then means that the parallely displaced vector $\xi^i[y]$ will be a functional of function space curves. If the affinely connected manifold is such that

$$B^i{}_{\alpha\beta\gamma}[y] = 0 \tag{4.31}$$

then the integrability conditions (4.23) will be satisfied identically and the solution (4.28) of the functional equation (4.20) or (4.21) will be independent of the choice of the curve.

5. *The functional curvature tensor and other affine functional invariants.* The law of transformation of an affine connection can obviously be written in the form

$$\begin{aligned}''y^i{}_{\alpha\beta}[\bar{y}] = \bar{\Gamma}^i{}_{\alpha\beta}[\bar{y}] + \bar{\Gamma}^\sigma{}_{\alpha\beta}\,'y_\sigma{}^i - \Gamma^i{}_{\alpha\beta}[y] - \Gamma^i{}_{\sigma\beta}\,'y_\alpha{}^\sigma \\ - \Gamma^i{}_{\alpha\sigma}\,'y_\beta{}^\sigma - \Gamma^i{}_{\sigma\tau}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau\end{aligned} \tag{5.1}$$

as is evident by a reference to formula (4.13). Since $''y_{\alpha\beta}{}^i$ is indeed a functional of $\bar{y}^\gamma$ we must have

$$'''y^i{}_{\alpha\beta\gamma}[\bar{y}] - '''y^i{}_{\alpha\beta\gamma}[\bar{y}] = 0. \tag{5.2}$$

To obtain this condition (which is obviously the condition for the self-adjointness of the differential $\delta''y_{\alpha\beta}{}^i$) in terms of the first functional derivatives $'y_\alpha{}^i[\bar{y}]$, we first calculate the third functional derivatives $'''y^i{}_{\alpha\beta\gamma}$ from (5.1) by taking the differential of both sides of (5.1) and equating the coefficients of $\delta\bar{y}^\gamma$. We thus shall have the condition (5.2) expressed in terms of $''y_{\alpha\beta}{}^i$ and $'y_\alpha{}^i$. If then we eliminate $''y_{\alpha\beta}{}^i$ from this final expression by means of (5.1) itself, we shall obtain the equations

$$\begin{aligned}\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}] + \bar{B}^\sigma{}_{\alpha\beta\gamma}\,'y_\sigma{}^i[\bar{y}] = B^i{}_{\alpha\beta\gamma}[y] + B^i{}_{\sigma\beta\gamma}\,'y_\alpha{}^\sigma \\ + B^i{}_{\alpha\sigma\gamma}\,'y_\beta{}^\sigma + B^i{}_{\alpha\beta\sigma}\,'y_\gamma{}^\sigma \\ + B^i{}_{\sigma\tau\gamma}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau + B^i{}_{\sigma\beta\tau}\,'y_\alpha{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\alpha\sigma\tau}\,'y_\beta{}^\sigma\,'y_\gamma{}^\tau \\ + B^i{}_{\sigma\tau\nu}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau\,'y_\gamma{}^\nu\end{aligned} \tag{5.3}$$

where $B^i{}_{\alpha\beta\gamma}[y]$ is the functional that was defined by (4.24). The unqiue continuous solution $\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}]$ of the Fredholm integral equation (5.3) is

$$\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}] = \left(^i_{\alpha\beta\gamma}\right) + 'y_\sigma{}^i[y]\ \left(^\sigma_{\alpha\beta\gamma}\right), \tag{5.4}$$

where

$$\begin{aligned}\left(^i_{\alpha\beta\gamma}\right) = B^i{}_{\alpha\beta\gamma}[y] + B^i{}_{\sigma\beta\gamma}\,'y_\alpha{}^\sigma[\bar{y}] + B^i{}_{\alpha\sigma\gamma}\,'y_\beta{}^\sigma + B^i{}_{\alpha\beta\sigma}\,'y_\gamma{}^\sigma \\ B^i{}_{\sigma\tau\gamma}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau + B^i{}_{\sigma\beta\tau}\,'y_\alpha{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\alpha\sigma\tau}\,'y_\beta{}^\sigma\,'y_\gamma{}^\tau + B^i{}_{\sigma\tau\nu}\,'y_\alpha{}^\sigma\,'y_\beta{}^\tau\,'y_\gamma{}^\nu.\end{aligned} \tag{5.5}$$

Another useful form of the law of transformation of $B^i{}_{\alpha\beta\gamma}$ is obtained by multiplying both sides of (5.3) by $\bar{y}_i{}^{\omega}[y]$ and integrating over the variable i. If we simplify the resultant expression by means of (3.6) we obtain

$$\bar{B}^i{}_{\alpha\beta\gamma}[\bar{y}]\ {}'y_i{}^{\omega}[\bar{y}] = -{}'\bar{y}_{\sigma}{}^{\omega}[y]\ \begin{pmatrix}\sigma\\ \alpha\beta\gamma\end{pmatrix}. \tag{5.6}$$

Formula (5.3) shows that $B^i{}_{\alpha\beta\gamma}[y]$ is a functional tensor which is covariant of order three and contravariant of order one.

By a process similar to the one by means of which we deduced (5.3) from (5.1), we can obtain the law of transformation of another functional tensor. In fact on taking the differential of both sides of (5.3), equating the coefficient of the differential $\delta\bar{y}^{\delta}$, and eliminating ${}''y^i{}_{\alpha\beta}[\bar{y}]$ by means of (5.1) we obtain the law of transformation of the functional tensor $B^i{}_{\alpha\beta\gamma;\delta}[y]$ defined by

$$B^i{}_{\alpha\beta\gamma;\delta}[y] = {}'B^i{}_{\alpha\beta\gamma,\delta}[y] + B^{\omega}{}_{\alpha\beta\gamma}\,\Gamma^i{}_{\omega\delta} - B^i{}_{\omega\beta\gamma}\,\Gamma^{\omega}{}_{\alpha\delta} - B^i{}_{\alpha\omega\gamma}\,\Gamma^{\omega}{}_{\beta\delta} - B^i{}_{\alpha\beta\omega}\,\Gamma^{\omega}{}_{\gamma\delta}. \tag{5.7}$$

We shall call $B^i{}_{\alpha\beta\gamma;\delta}$ the *covariant functional derivative of* $B^i{}_{\alpha\beta\gamma}$. *To distinguish this type of functional differentiation from the ordinary functional differentiation we shall use a semi-colon immediately preceding the index added on account of a covariant functional differentiation.*

It is clear now how one can proceed to deduce the infinite sequence of functional tensors

$$B^i{}_{\alpha\beta\gamma;\,\delta;\,\epsilon}[y], \quad \cdots, \quad B^i{}_{\alpha\beta\gamma;\,\delta;\,\epsilon;\,\ldots;\,\lambda}[y], \quad \cdots.$$

The general functional operation that yields a functional tensor of one higher covariant order than the functional tensor on which it operates will be called *covariant functional differentiation.* It is always possible to obtain the result of covariant functional differentiation by the processes of ordinary functional differentiation and elimination by means of (5.1). We shall see in § 8 that covariant functional differentiation is a special case of a more general functional operation that yields functional tensors of higher orders than the functional tensor on which it operates.

For the sake of future reference, we shall write down the *formula giving the covariant functional derivative of a functional tensor* $T^{ij\ldots k}_{ab\ldots c}$

$$\begin{aligned} T^{ij\ldots k}_{ab\ldots c;\,d}[y] &= {}'T^{ij\ldots k}_{ab\ldots c,\,d}[y] + T^{\omega j\ldots k}_{ab\ldots c}\,\Gamma^i{}_{\omega\delta} + T^{i\omega\ldots k}_{ab\ldots c}\,\Gamma^j{}_{\omega d} \\ &+ \cdots + T^{ij\ldots\omega}_{ab\ldots c}\,\Gamma^k{}_{\omega d} - T^{ij\ldots k}_{\omega b\ldots c}\,\Gamma^{\omega}{}_{ad} - T^{ij\ldots k}_{a\omega\ldots c}\,\Gamma^{\omega}{}_{bd} \\ &- \cdots - T^{ij\ldots k}_{ab\ldots\omega}\,\Gamma^{\omega}{}_{cd}. \end{aligned} \tag{5.8}$$

In (5.8), the functional $\Gamma^i{}_{a\beta}[y]$ is the affine connection of our affinely connected function space manifold. Of course it is understood that the differential $\overset{ij\ldots k}{ab\ldots c}$ is of the Volterra form

$$\delta T^{ij\ldots k}_{ab\ldots c} = {}'T^{ij\ldots k}_{ab\ldots c;\,d}\ \delta y^d \tag{5.9}$$

and the functional derivative ${}'T^{ij\ldots k}_{ab\ldots c;\,d}$ is continuous both as a functional of y^i and as a function of $ij\cdots kab\cdots cd$.

The functional operation of covariant differentiation yields a tensor when it operates on a tensor. The question arises as to the existence of *functional operations which yield a tensor when the operation is applied on a functional which has a definite law of transformation that is not the tensor law. We shall see later that such functional operations are of great importance in the theory of Riemannian function space manifolds* (see § 10).

To make these statements more definite let us consider a covariant functional tensor $\xi_i[y]$ with a differential

$$\delta\xi_i[y] = \eta_{(i)}[y]\ \delta y^i + {}'\xi_{i,a}[y]\ \delta y^a \quad \text{(the variable } i \text{ is not an integration variable).} \tag{5.10}$$

In (5.10), the functional $\eta_{(i)}[y]$ is of necessity a scalar functional, i. e.,

$$\bar{\eta}_{(i)}[\bar{y}] = \eta_{(i)}[y]. \tag{5.11}$$

The law of transformation of ${}'\xi_{a,\beta}[y]$

$$\begin{aligned}{}'\xi_{a,\beta}[\bar{y}] = {}'\xi_{a,\beta}[y] &+ {}'\xi_{\lambda,\beta}{}'y_a{}^\lambda[\bar{y}] + {}'\xi_{a,\lambda}{}'y_\beta{}^\lambda + {}'\xi_{\lambda,\mu}{}'y_a{}^\lambda\,{}'y_\beta{}^\mu + \xi_\lambda{}''y^\lambda{}_{a\beta}[\bar{y}] \\ &+ \eta_{(\alpha)}{}'y_\beta{}^\alpha + \eta_{(\beta)}{}'y_a{}^\beta + \eta_{(\lambda)}{}'y_a{}^\lambda\,{}'y_\beta{}^\lambda\end{aligned} \tag{5.12}$$

(α and β are not integration variables).

Hence by elimination of ${}''y^\lambda{}_{a\beta}$, we obtain the relation

$$\begin{aligned}\xi_{a;\beta}[\bar{y}] = \xi_{a;\beta}[y] &+ \xi_{\lambda;\beta}{}'y_a{}^\lambda + \xi_{a;\lambda}{}'y_\beta{}^\lambda + \xi_{\lambda;\mu}{}'y_a{}^\lambda\,{}'y_\beta{}^\mu \\ &+ \eta_{(\alpha)}[y]\,{}'y_\beta{}^\alpha + \eta_{(\beta)}{}'y_a{}^\beta + \eta_{(\lambda)}{}'y_a{}^\lambda\,{}'y_\beta{}^\lambda\end{aligned} \tag{5.13}$$

(α and β are not integration variables)

where

$$\begin{aligned}\xi_{a;\beta}[y] &= {}'\xi_{a,\beta}[y] - \Gamma^\mu{}_{a\beta}[y]\ \xi_\mu[y] \\ \bar{\xi}_{a;\beta}[\bar{y}] &= {}'\bar{\xi}_{a,\beta}[\bar{y}] - \Gamma^\mu{}_{a\beta}[\bar{y}]\ \bar{\xi}_\mu[\bar{y}].\end{aligned} \tag{5.14}$$

The following theorem is immediate.

THEOREM (5.I). *A necessary and sufficient condition that the covariant functional derivative of a vector $\xi_i[y]$ with a differential* (5.10) *be a tensor*

is that the scalar functional $\eta_i[y]$ *be identically zero.* Similar theorems can be proved for functional tensors of any order.

In case

$$\eta_{(i)}[y] \equiv 1 \tag{5.15}$$

we see from (5.13) and (2.30) that under the group of functional transformations, a sub-group of the transformations $\mathfrak{G}_s$, that induce an orthogonal Fredholm group in the differentials δy^i, and under that alone, does $\xi_{a;\beta}[y]$ transform like a tensor.

Let us assume now that

$$\delta\, '\xi_{i,a}[y] = ''\xi_{i,a\beta}\delta y^\beta \tag{5.16}$$

and that (5.15) holds. It is then a consequence of (5.14) that

$$\delta\, \xi_{a;\beta}[y] = '\xi_{a;\beta,\gamma}[y]\ \delta y^\gamma \tag{5.17}$$

The functional $\xi_{a\beta\gamma}[y]$ defined by

$$\xi_{a\beta\gamma}[y] = '\xi_{a;\beta,\gamma}[y] - \xi_{\lambda;\beta}[y]\ \Gamma^\lambda{}_{a\gamma}[y] - \xi_{a;\lambda}\,\Gamma^\lambda{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\beta{}_{a\gamma} \tag{5.18}$$

is a covariant functional tensor of order three. This result can be deduced from (5.13) by a method similar to the one used in arriving at (5.13) from the vector law of transformation of $\xi_i[y]$. We have thus derived a functional tensor $\xi_{a\beta\gamma}[y]$ by applying a more general functional operation than covariant functional differentiation to a functional which is not a tensor. The reader will see that a multitude of similar results can be obtained by starting with a general tensor instead of a covariant vector ξ_i.

The functional curvature tensor $B^i{}_{a\beta\gamma}[y]$ and its successive covariant derivatives are fixed functionals of the affine connection $\Gamma^i{}_{a\beta}$ and the functional derivatives of the affine connection. This sequence of tensors may be called affine functional tensor invariants.

DEFINITION. *A functional (not necessarily a tensor)*

$$F^{ij\ldots l}_{ab\ldots c}\ [\Gamma, {}'\Gamma, \cdots, {}^{(r)}\Gamma]$$

of the affine connection $\Gamma^i{}_{a\beta}$ *and its functional derivatives up to and including the rth will be said to be an affine functional invariant of order r and weight w if under the group of transformations* $\mathfrak{G}_s$

(1) *there is a definite law connecting its expression in the variable* $\bar{y}^i$ *in terms of its expression in the variable* y^i

(2) *the form of the functional remains unchanged when the functional arguments are taken to be* $\Gamma^i{}_{a\beta}$, ${}'\Gamma^i{}_{a\beta,\gamma_1}$, $\cdots$, ${}^{(r)}\Gamma^i{}_{a\beta,\gamma_1\ldots\gamma_r}$

(3) *the wth power of the Fredholm determinant* $D['y_\alpha{}^i]$ *of the functional derivative* $'y_\alpha{}^i[\bar{y}]$ *of a transformation* $\mathfrak{G}_s$ *is a factor of*

$$F^{ij\ldots l}_{ab\ldots c}\ [\bar{\Gamma}, {}'\bar{\Gamma}, \cdots, {}^{(r)}\bar{\Gamma}].$$

Clearly the affine connection $\Gamma^i{}_{\alpha\beta}[y]$ is an absolute (weight $w = 0$) affine functional invariant of order zero and the curvature tensor $B^i{}_{\alpha\beta\gamma}$ is an absolute affine functional invariant of order one.

The definition (4.24) shows that $B^i{}_{\alpha\beta\gamma}[y]$ satisfies the functional identities

$$B^i{}_{\alpha\beta\gamma} + B^i{}_{\gamma\alpha\beta} + B^i{}_{\beta\gamma\alpha} = 0, \tag{5.19}$$
$$B^i{}_{\alpha\beta\gamma} = -B^i{}_{\alpha\gamma\beta}.$$

6. *The fundamental integro-differential equation of affinely connected function space manifolds.* Consider a continuous functional $\Lambda^i{}_{\alpha\beta}[y^\lambda]$ that depends in particular on the variables i, α and β and is a symmetric function of α and β. Let $M^i{}_\alpha[y^\lambda]$ stand for a continuous functional that depends in particular on the variables i and α. Form now the non-linear integro-differential equation

$$\frac{\partial t^2}{\partial^2 y^i(t)} + \Lambda^i{}_{\alpha\beta}[y^\lambda(t)]\ \frac{\partial y^\alpha(y)}{\partial t}\ \frac{\partial y^\beta(t)}{\partial t} + M_\alpha{}^i[y^\lambda(t)]\left(\frac{\partial y^\alpha(t)}{\partial t}\right)^2 = 0. \tag{6.1}$$

In (6.1) we understand that the functional *operations* $\Lambda^i{}_{\alpha\beta}[\ ]$ and $M^i{}_\alpha[\ ]$ are independent of the parameter t. We wish to determine the conditions under which equation (6.1) is left invariant in form by the group of transformations $\mathfrak{G}_s$. Let us make a change of variable

$$y^i = y^i[\bar{y}^\lambda]$$

so that we have

$$y^i(t) = y^i[\bar{y}^\lambda(t)] \tag{6.2}$$

in which $\bar{y}^\lambda(t)$ is continuous in λ and analytic in t. We shall assume that the functional operation $y^i[\ ]$ yields an analytic function of t when it operates on $\bar{y}^\lambda(t)$ and eliminates the variable λ. Obviously

$$\bar{y}^\lambda(t+\epsilon) - \bar{y}^\lambda(t) = \epsilon\ [(\partial\bar{y}^\lambda/\partial t) + \sigma^\lambda(t)]$$

where

$$\lim_{\epsilon\to 0} \sigma^\lambda(t) = 0.$$

Hence from the properties of the Fréchét differentials of y^i it follows that

$$\frac{\partial y^i(t)}{\partial t} = \frac{\partial \bar{y}^i(t)}{\partial t} + {}'y_\alpha{}^i[\bar{y}^\lambda(t)] \frac{\partial \bar{y}^\alpha(t)}{\partial t}$$

(6.3)

$$\frac{\partial^2 y^i(t)}{\partial t^2} = \frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + {}''y^i{}_{\alpha\beta}[\bar{y}^\lambda(t)] \frac{\partial \bar{y}^\alpha(t)}{\partial t} \frac{\partial \bar{y}^\beta(t)}{\partial t} + {}'y_\alpha{}^i[\bar{y}^\lambda(t)] \frac{\partial^2 \bar{y}^\alpha(t)}{\partial t^2}.$$

A substitution in (6.1) of the expressions for $\partial y^i(t)/\partial t$ and $\partial^2 y^i(t)/\partial t^2$ as given by (6.3) gives

$$\frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + {}'y_\alpha{}^i[\bar{y}] \frac{\partial^2 \bar{y}^\alpha(t)}{\partial t^2} = \theta^i(t) \tag{6.4}$$

where

$$\theta^i(t) = -\left({}''y^i{}_{\alpha\beta}[\bar{y}] + \Lambda^i{}_{\alpha\beta} + \Lambda^i{}_{\lambda\beta}{}'y_\alpha{}^\lambda + \Lambda^i{}_{\alpha\lambda}{}'y_\beta{}^\lambda + \Lambda^i{}_{\lambda\mu}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\mu + \mathrm{M}_\alpha{}^i\,{}'y_\beta{}^\alpha + \mathrm{M}_\beta{}^i{}'y_\alpha{}^\beta + \mathrm{M}_\lambda{}^i{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\lambda\right) \frac{\partial \bar{y}^\alpha(t)}{\partial t} \frac{\partial \bar{y}^\beta(t)}{\partial t} - \mathrm{M}_\alpha{}^i \left(\frac{\partial \bar{y}^\alpha(t)}{\partial t}\right)^2.$$

The solution of the Fredholm integral equation (6.4) is

$$\frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + \bar{\Lambda}^i{}_{\alpha\beta}[\bar{y}^\lambda(t)] \frac{\partial \bar{y}^\alpha(t)}{\partial t} \frac{\partial \bar{y}^\beta(t)}{\partial t} + \bar{\mathrm{M}}_\alpha{}^i[\bar{y}^\lambda(t)] \left(\frac{\partial \bar{y}^\alpha(t)}{\partial t}\right)^2 = 0 \tag{6.5}$$

where

$$\bar{\Lambda}^i{}_{\alpha\beta}[\bar{y}^\lambda(t)] = \left[{}^{i}_{\alpha\beta}\right] + {}'\bar{y}_\sigma{}^i[y] \left[{}^{\sigma}_{\alpha\beta}\right], \tag{6.6}$$

$$\left[{}^{i}_{\alpha\beta}\right] = {}''y^i{}_{\alpha\beta}[\bar{y}] + \Lambda^i{}_{\alpha\beta} + \Lambda^i{}_{\lambda\beta}{}'y_\alpha{}^\lambda + \Lambda^i{}_{\alpha\lambda}{}'y_\beta{}^\lambda + \Lambda^i{}_{\lambda\mu}{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\mu + \mathrm{M}_\alpha{}^i\,{}'y_\beta{}^\alpha + \mathrm{M}_\beta{}^i\,{}'y_\alpha{}^\beta + \mathrm{M}_\lambda{}^i\,{}'y_\alpha{}^\lambda\,{}'y_\beta{}^\lambda$$

and

$$\bar{\mathrm{M}}_\alpha{}^i[\bar{y}^\lambda(t)] = \mathrm{M}_\alpha{}^i + {}'\bar{y}_\sigma{}^i[y]\, \mathrm{M}_\alpha{}^\sigma[y^\lambda(t)]. \tag{6.7}$$

It is to be observed that in the expression for $\left[{}^{i}_{\alpha\beta}\right]$, the terms $\mathrm{M}_\alpha{}^i\,{}'y_\beta{}^\alpha$ and $\mathrm{M}_\beta{}^i\,{}'y_\alpha{}^\beta$ are to be understood in the light of an ordinary product—an exception to the integration convention for repeated indices. *Thus the form of equation* (6.1) *persists under a transformation* (6.2) *if and only if the functionals* $\Lambda^i{}_{\alpha\beta}$ *and* $\mathrm{M}_\alpha{}^i$ *transform according to the rules* (6.6) *and* (6.7) *respectively.*

In this paper we shall direct our attention mainly to equations (6.1) for which

$$\mathrm{M}_\alpha{}^i = 0,$$

a condition which persists under a change of variable y^i. The following important theorem is now immediate.

THEOREM (6.1). *A necessary and sufficient condition that the form of the integro-differential equation*

$$\frac{\partial^2 y^i(t)}{\partial t^2} + \Lambda^i{}_{\alpha\beta}[y^\lambda(t)] \frac{\partial y^\alpha(t)}{\partial t} \frac{\partial y^\beta(t)}{\partial t} = 0 \tag{6.8}$$

persist under a transformation of the variable y^i is that the functional $\Lambda^i{}_{\alpha\beta}$ have the law of transformation of an affine connection.

The solution of equation (6.8), if it exists, defines a system of parameterized curves with a parameterization t in our general function space. Let us consider an affinely connected function space manifold determined by the affine connection $\Gamma^i{}_{\alpha\beta}[y]$. For different functionals $\Lambda^i{}_{\alpha\beta}$, equations (6.8) give rise to various systems of curves in this affinely connected function space manifold. The particular system of parameterized curves which satisfy the equation

$$\frac{\partial^2 y^i(t)}{\partial t} + \Gamma^i{}_{\alpha\beta}[y^\lambda(t)] \frac{\partial y^\alpha(t)}{\partial t} \frac{\partial y^\beta(t)}{\partial t} = 0 \tag{6.9}$$

will be called the parameterized *geodesics* or the parameterized *paths* of our affinely connected function space manifold.

To obtain a formal solution of the integro-differential equation (6.9) we write down the Taylor development

$$y^i(t) = y^i(t_0) + \left(\frac{\partial y^i(t)}{\partial t}\right)_{t=t_0} (t - t_0) + (1/2!) \left(\frac{\partial^2 y^i(t)}{\partial t^2}\right)_{t=t_0} (t - t_0)^2 + \cdots + \cdots \tag{6.10}$$

For initial conditions let us take

$$\begin{aligned} q^i &= y^i(t_0) \\ p^i &= \left(\frac{\partial y^i(t)}{\partial t}\right)_{t=t_0} \end{aligned} \tag{6.11}$$

where q^i and p^i are arbitrary continuous functions of the real variable i. One observes that (6.9) gives the derivative $\left(\frac{\partial^2 y^i}{\partial t^2}\right)_{t=t_0}$ as a homogeneous quadratic functional of p^i. Hence the mth derivative of y^i taken for $t = t_0$ will be a homogeneous functional polynomial of degree m in p^i. Thus a formal solution of (6.9) that takes on the initial conditions (6.11) is given by the development

$$y^i(t) = q^i + p^i(t - t_0) - (1/2!)\, \Gamma^i{}_{\alpha\beta}[q^\lambda]\, p^\alpha p^\beta (t - t_0)^2 - (1/3!) \Gamma^i{}_{\alpha\beta\gamma}[q^\lambda]\, p^\alpha p^\beta p^\gamma (t - t_0)^3 - \cdots \tag{6.12}$$

where

$$(6.13)\quad \Gamma^{i}{}_{\alpha\beta\gamma}[q^{\lambda}] = (1/3)P('\Gamma^{i}{}_{\alpha\beta,\gamma}[q^{\lambda}] - \Gamma^{i}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\alpha\gamma} - \Gamma^{i}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\gamma})$$

and in general

$$(6.14)\quad \Gamma^{i}_{\alpha\beta\gamma\ldots\mu\nu}[q^{\lambda}] = (1/N)P\,('\Gamma^{i}_{\alpha\beta\gamma\ldots\mu,\nu}[q^{\lambda}] - \Gamma^{i}_{\sigma\beta\gamma\ldots\mu}[q^{\lambda}]\,\Gamma^{\sigma}{}_{\alpha\nu}[q^{\lambda}] - \cdots - \Gamma^{i}_{\alpha\beta\gamma\ldots\sigma}\,\Gamma^{\sigma}{}_{\mu\nu}).$$

In (6.14), N denotes the number of variables $\alpha\,\beta\,\gamma\cdots\mu\,\nu$ and the symbol P stands for the operation that adds to the terms already in the parenthesis all terms obtainable from them by a cyclic permutation of the variables $\alpha\,\beta\,\gamma\cdots\mu\,\nu$.

The formal development in (6.12) is a power series in $(t - t_0)$

$$(6.15)\qquad a_0{}^{i} + a_1{}^{i}(t - t_0) + a_2{}^{i}(t - t_0)^2 + \cdots$$

in which the coefficients $a_0{}^{i}, a_1{}^{i}, \cdots$, are continuous functions of the real variable i $(a \leq i \leq b)$. The development in (6.12) will then be uniformly and absolutely convergent in i and t for

$$a \leq i \leq b, \qquad t_0 \leq t \leq t_1$$

if $|\,t_1 - t_0\,|$ is sufficiently small and the affine connection with its functional derivatives obeys certain obvious inequalities. Under these restrictions then, the formal development on the right side of (6.12) will represent a function analytic in t and continuous in i for t and i ranging over the respective intervals (t_0, t_1) and (a, b). This function $y^{i}(t)$ will then be the unique solution, analytic in t and continuous in i, of (6.9) under the initial conditions (6.11).

7. *Geodesic and normal functional coördinates.* The analysis in the theory of the affinely connected manifold can often be simplified by the introduction of suitable canonical variables. More definitely, the introduction of a functional coördinate system $\bar{y}^{i}$ such that

$$(7.1)\qquad (\Gamma^{i}{}_{\alpha\beta}[\bar{y}^{\lambda}])_{\bar{y}^{\lambda}=0} = 0$$

is frequently useful. We shall call such a system of functional coördinates $\bar{y}^{i}$ a *geodesic* system. The *origin* of this geodesic system of coördinates is the point P whose coördinates are $\bar{y}^{\lambda} = 0$. To show the existence of a geodesic system for each point P of our affinely connected manifold, let us consider the functional transformation

$$(7.2)\qquad y^{i} = q^{i} + \bar{y}^{i} - \tfrac{1}{2}\,\Gamma^{i}{}_{\alpha\beta}[q^{\lambda}]\,\bar{y}^{\alpha}\bar{y}^{\beta}$$

in which q^i is a given continuous function of i and represents the coördinates of a given point P in the y^i coördinate system. Clearly the set of values

$$y^i = q^i, \qquad \bar{y}^i = 0 \tag{7.3}$$

satisfy the functional equation (7.2). Assuming that $\Gamma^i{}_{\alpha\beta}[q^\lambda]$ is such that (7.2) is a transformation $\mathfrak{G}_s$, we see that (7.2) is a transformation of coördinates such that

$$({}'y_a{}^i\,[\bar{y}])_{\bar{y}^\lambda=0} = 0, \qquad ({}''y_a{}^i[\bar{y}])_{\bar{y}^\lambda=0} = -\,\Gamma^i{}_{\alpha\beta}\,[q^\lambda]. \tag{7.4}$$

Hence (7.1) follows from (5.1) by means of the relations (7.3) and (7.4). Thus (7.2) defines a geodesic coördinate system $\bar{y}^i$ with $\bar{y}^i = 0$ as origin. Such a system $\bar{y}^i$ is, however, merely one of an infinitude of geodesic systems; for, any transformation $\mathfrak{G}_s$ that satisfies conditions (7.4) is a geodesic system.

An arbitrary transformation on the general coördinates y^i induces in general a complicated transformation in the geodesic variables $\bar{y}^i$. There is a particular type of geodesic coördinates, however, which suffer merely a Fredholm transformation.

If in equation (6.12), we put

$$z^i = p^i\,(t - t_0) \tag{7.5}$$

we shall have the equation

$$y^i = q^i + z^i - (1/2\,!)\Gamma^i{}_{\alpha\beta}[q^\lambda]\,z^\alpha z^\beta - (1/3\,!)\Gamma^i{}_{\alpha\beta\gamma}[q^\lambda]\,z^\alpha z^\beta z^\gamma - \cdots - \cdots. \tag{7.6}$$

If the affine connection is suitably restricted, equation (7.6) will be a transformation $\mathfrak{G}_\infty$ with an inverse

$$\begin{aligned} z^i = y^i - q^i &+ (1/2\,!)a^i{}_{\alpha\beta}(y^\alpha - q^\alpha)\,(y^\beta - q^\beta) \\ &+ (1/3\,!)a^i{}_{\alpha\beta\gamma}(y^\alpha - q^\alpha)\,(y^\beta - q^\beta)\,(y^\gamma - q^\gamma) + \cdots. \end{aligned} \tag{7.7}$$

The coördinates z^i defined by (7.6) are a particular kind of geodesic coördinates. Such geodesic coördinates will be called *normal coördinates.* Equation (7.5) for a fixed function p^i gives then the equation of a path (solution of (6.9)) through the origin $y^i = q^i$ (i. e., $z^i = 0$) of the normal coördinate system z^i defined by (7.6).

A functional transformation

$$\bar{y}^i = \bar{y}^i[y] \tag{7.8}$$

transforms the equations of the paths

$$y^i = y^i(t) \tag{7.9}$$

into

$$\bar{y}^i = \bar{y}^i(t). \tag{7.10}$$

The transformed equations (7.10) of the paths satisfy the integro-differential equation

$$\frac{\partial^2 \bar{y}^i(t)}{\partial t^2} + \Gamma^i{}_{\alpha\beta}[\bar{y}^\lambda(t)] \frac{\partial \bar{y}^\alpha(t)}{\partial t} \frac{\partial \bar{y}^\beta(t)}{\partial t} = 0. \tag{7.11}$$

The transformation (7.8) changes the initial conditions (6.11) for equation (6.9) into

$$\bar{q}^i = \bar{y}^i(t_0) = \bar{q}^i[q^\lambda] \tag{7.12}$$
$$\bar{p}^i = \left(\frac{\partial \bar{y}^i(t)}{\partial t}\right)_{t=t_0}$$

for the transformed equation (7.11). Since

$$\left(\frac{\partial \bar{y}^i(t)}{\partial t}\right)_{t=t_0} = \left(\frac{\partial y^i(t)}{\partial t} + {}'\bar{y}_\alpha{}^i\,[y^\lambda(t)] \frac{\partial y^\alpha(t)}{\partial t}\right)_{t=t_0} \tag{7.13}$$

it follows that

$$\bar{p}^i = p^i + {}'\bar{y}_\alpha{}^i[q^\lambda]\, p^\alpha. \tag{7.14}$$

Multiplying both sides of (7.14) by $(t - t_0)$ we obtain

$$\bar{z}^i = z^i + {}'\bar{y}_\alpha{}^i[q^\lambda]\, z^\alpha. \tag{7.15}$$

Since q^λ is a fixed function, it follows that (7.15) is a Fredholm transformation. Hence we have the important theorem.

THEOREM (7.I). *An arbitrary transformation $\mathfrak{G}_\infty$ in the coördinates y^i induces a Fredholm transformation in the normal coördinates z^i which are determined by the coördinates y^i and a point P with coördinates $y^i = q^i$.*

Let $\mathrm{N}^i{}_{\alpha\beta}[z]$ be the expression for the affine connection in a normal coördinate system z^i. Then since (7.5) must satisfy

$$\frac{\partial^2 z^i(t)}{\partial t^2} + \mathrm{N}^i{}_{\alpha\beta}[z^\lambda(t)] \frac{\partial z^\alpha(t)}{\partial t} \frac{\partial z^\beta(t)}{\partial t} = 0 \tag{7.16}$$

we see that

$$\mathrm{N}^i{}_{\alpha\beta}[z^\lambda(t)] p^\alpha p^\beta = 0. \tag{7.17}$$

Consequently we have the result

$$\mathrm{N}^i{}_{\alpha\beta}[o] = 0. \tag{7.18}$$

8. *A generalization of covariant functional differentiation.* In § 5 we found by means of elimination processes a functional operation, covariant functional differentiation, which produced a functional tensor of one higher order than the tensor on which it operated. Furthermore we saw in the same paragraph that a slightly different functional operation from covariant func-

tional differentiation gave rise to a functional tensor when it operated on a functional with a law of transformation (5.13) in which $\eta_{(\alpha)} = 1$. It is the purpose of this paragraph to show that functional tensors of any order can be built up by means of functional operations which include those of § 5 as a special case. We shall use the theory of normal coördinates. It should be observed, however, that the final results *do not* depend for their validity on the actual existence or non-existence of normal coördinates—even if the series (7.6) does not converge we can use the formal theory of normal coördinates as a *heuristic guide* to obtain the final results, the validity of which can be established by the elimination processes of § 5.

Let $f^{\iota}{}_{\alpha\beta}[y^{\lambda}]$ be a continuous functional of y^{λ} that has a differential of the Volterra form

$$\delta f^{\iota}{}_{\alpha\beta}[y^{\lambda}] = {}'f^{\iota}{}_{\alpha\beta,\gamma}[y^{\lambda}]\,\delta y^{\gamma} \tag{8.1}$$

and a definite law of transformation

$$\begin{aligned}\bar{f}^{\iota}{}_{\alpha\beta}[\bar{y}] = {} & f^{\iota}{}_{\alpha\beta}[y] + f_{\sigma}{}^{\iota}{}_{\beta}\,{}'y_{\alpha}{}^{\sigma}[\bar{y}] + f^{\iota}{}_{\alpha\sigma}\,{}'y_{\beta}{}^{\sigma} + f^{\iota}{}_{\sigma\tau}\,{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\tau} + \rho^{\iota}{}_{\alpha\beta} \\ & + {}'\bar{y}_{\lambda}{}^{\iota}(f^{\lambda}{}_{\alpha\beta} + f^{\lambda}{}_{\sigma\beta}\,{}'y_{\alpha}{}^{\sigma} + f^{\lambda}{}_{\alpha\sigma}\,{}'y_{\beta}{}^{\sigma} + f^{\lambda}{}_{\sigma\tau}\,{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\tau} + \rho^{\lambda}{}_{\alpha\beta}).\end{aligned} \tag{8.2}$$

The quantity $\rho^{\iota}{}_{\alpha\beta}$ in (8.2) is assumed to be independent of any normal coördinates z^{ι} whenever it is referred to such a system of coördinates. Let $\phi^{\iota}{}_{\alpha\beta}[z^{\lambda}]$ be the expression for $f^{\iota}{}_{\alpha\beta}$ when it is referred to a system of normal coördinates z^{ι} which is determined by the arbitrary coördinates y^{ι} and a point P with coördinates q^{ι}. By calculation we obtain

$$'\bar{\phi}^{\iota}{}_{\alpha\beta,\gamma}[z] = \left(\begin{smallmatrix}\iota\\ \alpha\beta\gamma\end{smallmatrix}\right) + {}'\bar{z}_{\lambda}{}^{\iota}\left(\begin{smallmatrix}\lambda\\ \alpha\beta\gamma\end{smallmatrix}\right) \tag{8.3}$$

in which

$$\begin{aligned}\left(\begin{smallmatrix}\iota\\ \alpha\beta\gamma\end{smallmatrix}\right) = {} & {}'\phi^{\iota}{}_{\alpha\beta,\gamma}[z] + {}'\phi^{\iota}{}_{\sigma\beta,\gamma}\,{}'z_{\alpha}{}^{\sigma} + {}'\phi^{\iota}{}_{\alpha\sigma,\gamma}\,{}'z_{\beta}{}^{\sigma} + {}'\phi^{\iota}{}_{\alpha\beta,\sigma}\,{}'z_{\gamma}{}^{\sigma} + {}'\phi^{\iota}{}_{\alpha\sigma,\tau}\,{}'z_{\beta}{}^{\sigma}\,{}'z_{\gamma}{}^{\tau} \\ & + {}'\phi^{\iota}{}_{\sigma\beta,\tau}\,{}'z_{\alpha}{}^{\sigma}\,{}'z_{\gamma}{}^{\tau} + {}'\phi^{\iota}{}_{\sigma\tau,\gamma}\,{}'z_{\alpha}{}^{\sigma}\,{}'z_{\beta}{}^{\tau} + {}'\phi^{\iota}{}_{\sigma\tau,\nu}\,{}'z_{\alpha}{}^{\sigma}\,{}'z_{\beta}{}^{\tau}\,{}'z_{\gamma}{}^{\nu}.\end{aligned}$$

From the arbitrariness of the point P with coördinates $y^{\iota} = q^{\iota}$, from (8.3) and from (7.15) we see that the functional $f^{\iota}{}_{\alpha\beta;\gamma}[y]$ of y^{λ} defined by

$$f^{\iota}{}_{\alpha\beta;\gamma} = ({}'\phi^{\iota}{}_{\alpha\beta,\gamma}[z])_{z^{\lambda}=0} \tag{8.4}$$

is a functional tensor covariant of order three and contravariant of order one.

The explicit form of the functional $f^{\iota}{}_{\alpha\beta;\gamma}[y]$ in terms of the affine connection $\Gamma^{\iota}{}_{\alpha\beta}[y]$ and its functional derivatives of course depends on the quantity $\rho^{\iota}{}_{\alpha\beta}$ in (8.2). We shall treat presently a few important cases. For the case

$$\rho^{\iota}{}_{\alpha\beta} = {}''y^{\iota}{}_{\alpha\beta}[\bar{y}] \tag{8.5}$$

we have

$$(8.6)\qquad \phi^i{}_{a\beta}[z] + \phi^\sigma{}_{a\beta}[z]\,'y_\sigma{}^i[z] = {}''y^i{}_{a\beta}[z] + f^i{}_{a\beta}[y] + f^i{}_{\sigma\beta}\,'y_a{}^\sigma[z] + f^i{}_{a\sigma}\,'y_\beta{}^\sigma + f^i{}_{\sigma\tau}\,'y_a{}^\sigma\,'y_\beta{}^\tau.$$

Hence

$$(8.7)\qquad {}'\phi^i{}_{a\beta,\gamma}[z] = {}'''y^i{}_{a\beta\gamma}[z] + {}'f^i{}_{a\beta,\gamma}[y] + f^i{}_{\sigma\beta}\,''y^\sigma{}_{a\gamma}[z] + f^i{}_{a\sigma}\,''y^\sigma{}_{\beta\gamma} - \phi^\sigma{}_{a\beta}\,''y^i{}_{\sigma\gamma}[z] + *$$

where the $*$ stands for terms which vanish with ${}'y_a{}^\sigma[z]$. Formula (7.6) shows that at the origin of normal coördinates z^i we have

$$(8.8)\qquad {}'y_a{}^\sigma = 0;\quad {}''y^\sigma{}_{a\beta} = -\Gamma^\sigma{}_{a\beta};\quad {}'''y^\sigma{}_{a\beta\gamma} = -\Gamma^\sigma{}_{a\beta\gamma}.$$

With the aid of these formulae as well as (8.6) and (8.7) we finally obtain

$$(8.9)\qquad f^i{}_{a\beta;\gamma}[y] = {}'f^i{}_{a\beta,\gamma}[y] - \Gamma^i{}_{a\beta\gamma} - f^i{}_{\sigma\beta}\Gamma^\sigma{}_{a\gamma} - f^i{}_{a\sigma}\Gamma^\sigma{}_{\beta\gamma} + f^\sigma{}_{a\beta}\Gamma^i{}_{\sigma\gamma} - \Gamma^\sigma{}_{a\beta}\Gamma^i{}_{\sigma\gamma}.$$

It is easy to show that the case

$$\rho^i{}_{a\beta} \equiv 0$$

yields the covariant functional derivative $f^i{}_{a\beta;\gamma}[y]$ of the functional tensor $f^i{}_{a\beta}[y]$.

Suppose now that a functional $f^{ij\cdots k}_{ab\cdots c}[y]$ has differentials of the Volterra form

$$(8.10)\qquad {}^{(a)}f^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_\alpha}[y]\;\delta y^{\lambda_1}\delta y^{\lambda_2}\cdots\delta^{\lambda_\alpha} \qquad (\alpha = 1, 2, \cdots, p)$$

in which the functional derivatives ${}^{(a)}f$ are continuous functionals of y^i and continuous functions of $ij\cdots k\,ab\cdots c\,\lambda_1\cdots\lambda_a$. Let $\phi^{ij\cdots k}_{ab\cdots c}[z]$ represent the expression for $f^{ij\cdots k}_{ab\cdots c}$ in a normal coördinate system z^i. Let us assume that the law of transformation of the functional $f^{ij\cdots k}_{ab\cdots c}[y]$ is such that ${}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_p}[y]$ transforms like a functional tensor under the Fredholm group (7.15) of the normal coördinates z^i. *The functional* $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\lambda_2\cdots\lambda_p}[y]$ *of* y^i *defined by*

$$(8.11)\qquad f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\lambda_2\cdots\lambda_p}[q] = \left({}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\,\lambda_1\lambda_2\cdots\lambda_p}[z]\right)_P$$

is a functional tensor with $ij\cdots k$ *as contravariant indices and* $a\cdots c\lambda_1\cdots\lambda_p$ *as covariant indices.—In* (8.11) *we understand that the functional* ${}^{(p)}\phi$ *is evaluated at the origin* P *of the normal coördinates* z^i. The functional $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\cdots\lambda_p}$ will be said to be the *pth extension* of the functional $f^{ij\cdots k}_{ab\cdots c}$. When $p = 1$ and $f^{ij\cdots k}_{ab\cdots c}[y]$ is a functional tensor, then $f^{ij\cdots k}_{ab\cdots c}$ is merely the covariant functional derivative of the functional tensor $f^{ij\cdots k}_{ab\cdots c}$. We ob-

serve that the pth successive covariant functional derivative $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\,;\,\lambda_2;\,\cdots\,;\,\lambda_p}$ is not in general the same as the pth extension of the functional $f^{ij\cdots k}_{ab\cdots c}$. The proof that the functional $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\cdots\lambda_p}$ defined by (8.11) is a functional tensor is obvious on the basis of our assumptions about the functional $f^{ij\cdots k}_{ab\cdots c}$. It is interesting to observe that if the differentials of the functional $f^{ij\cdots k}_{ab\cdots c}$ are not of the Volterra form [see (8.10)], the functional defined by (8.11) will not be in general a functional tensor. This can be seen most easily from the fact that the functional derivatives ${}^{(p)}\phi^{ij\cdots k}_{ab\cdots c,\,\lambda_1\cdots\lambda_p}[z]$ will not transform like functional tensors under the Fredholm group (7.15). As an example we may take the case of a covariant functional vector $\xi_a[y]$ with a differential (5.10). The functional $\xi_{a;\beta}[y]$ defined by the process (8.11) will obey the non-tensor law (5.13).

Any functional $f^{ij\cdots k}_{ab\cdots c;\,\lambda_1\cdots\lambda_p}$ defined by (8.11) can be expressed as functional of $f^{ij\cdots k}_{ab\cdots c}$, $\Gamma^i{}_{a\beta}$, and their functional derivatives. The procedure is similar to the simple case (8.9) already treated.

9. *Normal functional tensors and replacement theorems.* The curvature tensor $B^i{}_{a\beta\gamma}$ and its successive covariant functional derivatives are, as we have seen, a sequence of affine functional invariants. There is, however, another important sequence of tensors each of which is an affine functional invariant. These affine functional invariants $A^i_{a\beta\gamma_1\cdots\gamma_p}[y]$ are defined by

$$A^i_{a\beta\gamma_1\cdots\gamma_p} = A^i_{a\beta;\,\gamma_1\cdots\gamma_p} \tag{9.1}$$

where

$$A^i_{a\beta;\,\gamma_1\cdots\gamma_p}[q] = \left({}^{(p)}N^i_{a\beta,\gamma_1\cdots\gamma_p}[z]\right)_P . \tag{9.2}$$

In (9.2), we understand that $N^i{}_{a\beta}[z]$ is the affine connection $\Gamma^i{}_{a\beta}[y]$ referred to a system of normal coördinates z^i with origin at the point P. It follows from the discussion of the last paragraph that the functionals $A^i_{a\beta\gamma_1\cdots\gamma_p}[y]$ are functional tensors covariant of order $p+2$ and contravariant of order one. We shall call the functional $A^i_{a\beta\gamma_1\cdots\gamma_p}[y]$ the *pth normal affine functional tensor.*

Let us assume now that the affine connection $N^i{}_{a\beta}[z]$ possesses a uniformly and absolutely convergent Volterra development about the functional neighborhood of the origin P of the normal coördinates z^i. Since $N^i{}_{a\beta}$ vanishes at the origin of normal coördinates, it follows that

$$N^i{}_{a\beta}[z] = A^i{}_{a\beta\gamma}[q]\,z^\gamma + (1/2!)A^i{}_{a\beta\gamma\delta}[q]\,z^\gamma z^\delta + (1/3!)A^i{}_{a\beta\gamma\delta\epsilon}z^\gamma z^\delta z^\epsilon + \cdots + \cdots . \tag{9.3}$$

Hence the sequence of normal functional tensors A constitutes a *complete set of affine functional invariants* in the sense that the affine connection $N^i{}_{\alpha\beta}[z]$ is completely determined by them, which in turn determines the affinely connected manifold.

Equation (7.17) shows that

$$N^i{}_{\alpha\beta}[z]z^\alpha z^\beta = 0. \tag{9.4}$$

Consequently

$$A^i{}_{\alpha\beta\gamma}z^\alpha z^\beta z^\gamma + (1/2!)\, A^i{}_{\alpha\beta\gamma\delta}z^\alpha z^\beta z^\gamma z^\delta + \cdots = 0. \tag{9.5}$$

The functional derivatives of (9.5) must vanish identically in the arbitrary function z^i. In particular they must vanish at the origin $z^i = 0$. Hence

$$A^i{}_{\alpha\beta\gamma}[y] + A^i{}_{\gamma\alpha\beta} + A^i{}_{\beta\gamma\alpha} = 0 \tag{9.6}$$

and in general

$$S(A^i{}_{\alpha\beta\gamma\ldots\omega}) = 0 \tag{9.7}$$

in which S stands for the operation that adds to the functional $A^i{}_{\alpha\beta\gamma\ldots\omega}$ all functionals obtained from it by a permutation of the variables $\alpha\beta\gamma\cdots\omega$. It is obvious from the definition (9.2) that

$$A^i{}_{\alpha\beta\gamma\ldots\omega} = A^i{}_{\beta\alpha\gamma\ldots\omega}, \tag{9.8}$$

$$A^i{}_{\alpha\beta\gamma\ldots\omega} = A^i{}_{\alpha\beta\lambda\ldots\rho}, \tag{9.9}$$

where $\lambda\cdots\rho$ is any permutation of the variables $\gamma\cdots\omega$.

As a special case of formula (8.9) we have

$$A^i{}_{\alpha\beta\gamma}[y] = {}'\Gamma^i{}_{\alpha\beta,\gamma}[y] - \Gamma^i{}_{\alpha\beta\gamma} - \Gamma^i{}_{\sigma\beta}\Gamma^\sigma{}_{\gamma\alpha} - \Gamma^i{}_{\sigma\alpha}\Gamma_{\beta\gamma}{}^\sigma. \tag{9.10}$$

To obtain the corresponding formula for the pth normal tensor one need only employ the same method that led to formula (8.9).

It is possible to express every affine tensor functional invariant as a functional of the normal tensors A alone. In fact let us consider an affine tensor invariant T of order r and weight w. We have

$$T^{ij\ldots l}_{ab\ldots c}\left[N^\sigma{}_{\alpha\beta}[z],\ {}'N^\sigma{}_{\alpha\beta,\gamma_1},\cdots,\ {}^{(r)}N^\sigma{}_{\alpha\beta,\gamma_1\ldots\gamma_r}\right] = (D[{}'y_\nu{}^\mu])^\omega\,(T[\Gamma[y],\cdots],\ {}'y). \tag{9.11}$$

In (9.11), $N^i{}_{\alpha\beta}[z]$ stands for the affine connection $\Gamma^i{}_{\alpha\beta}[y]$ referred to a system or normal coördinates z^i determined by the arbitrary coördinates z^i and an origin $y^i = q^i$. If we evaluate (9.11) at the origin $y^i = q^i$, we obtain the relation

$$T^{ij\ldots l}_{ab\ldots c}\left[0,\ A^\sigma{}_{\alpha\beta\gamma_1}[q],\cdots,\ \mathrm{A}^\sigma{}_{\alpha\beta\gamma_1\ldots\gamma_r}\right] = T^{ij\ldots l}_{ab\ldots c}\left[\Gamma^\sigma{}_{\alpha\beta}[q],\ {}'\Gamma^\sigma{}_{\alpha\beta,\gamma_1},\cdots,\ {}^{(r)}\Gamma^\sigma{}_{\alpha\beta,\gamma_1\ldots\gamma_r}\right]$$

since the Fredholm determinant $D[{}'y_\nu{}^\mu]$ has the value one at the origin of normal coördinates. The following theorem follows immediately.

THEOREM (9.I). *Any affine tensor functional invariant of order r (relative or absolute) can be put into the form*

$$T^{ij\cdots l}_{ab\cdots c}[0,\ A^\sigma{}_{a\beta\gamma_1},\ \cdots,\ A^\sigma{}_{a\beta\gamma_1\cdots\gamma_r}] \tag{9.12}$$

by replacing the affine connection $\Gamma^i{}_{a\beta}$ by zero and the functional derivatives of $\Gamma^i{}_{a\beta}$ by the corresponding affine normal functional tensors $A^i{}_{a\beta\gamma\cdots\delta}$.

Corresponding theorems may be proved for affine invariants that are not tensors. For example, if an affine functional invariant $f^i{}_{a\beta}$ of order r and weight w has the law of transformation

$$f^i{}_{a\beta}[\bar\Gamma^\sigma{}_{ij},\ \bar\Gamma^\sigma{}_{ij,k_1},\ \cdots,\ {}^{(r)}\bar\Gamma^\sigma{}_{ij,k_1\cdots k_r}] = (D[{}'y_\nu{}^\mu])^w\left[\left(\begin{matrix} i \\ a\beta \end{matrix}\right) + {}'\bar y_\lambda{}^i\left(\begin{matrix} \lambda \\ a\beta \end{matrix}\right)\right], \tag{9.13}$$

where

$$\left(\begin{matrix} i \\ a\beta \end{matrix}\right) = {}''y^i{}_{a\beta} + f^i{}_{a\beta}\,[\Gamma^\sigma{}_{ij},\ \cdots,\ {}^{(r)}\Gamma^\sigma{}_{ij,k_1\cdots k_r}] + f^i{}_{\rho\beta}{}'y_a{}^\rho + f^i{}_{a\rho}{}'y_\beta{}^\rho + f^i{}_{\rho\phi}{}'y_a{}^\rho\,{}'y_\beta{}^\phi,$$

it can be put in the form

$$f^i{}_{a\beta}\,[0,\ A^\sigma{}_{ijk_1},\ \cdots,\ A^\sigma{}_{ijk_1\cdots k_r}] - \Gamma^i{}_{a\beta}. \tag{9.14}$$

Furthermore, theorem (9.I) holds also for the case of an affine invariant $f_{a\beta}\,[\Gamma^i{}_{\gamma\delta},\ {}'\Gamma^i{}_{\gamma\delta,\epsilon_1},\ \cdots,\ {}^{(r)}\Gamma^i{}_{\gamma\delta,\epsilon_1\cdots\epsilon_r}]$ which transforms like the functional $\xi_{i;\beta}$ in (5.13).

Before we turn our attention to the study of a particular type of affinely connected manifold, it is worth while to observe that the problem of the *equivalence* of two affinely connected function space manifolds is a difficult problem in the theory of functional algebraic forms (see § 2) and in the theory of functional equations with functional derivatives.

10. *Quadratic differential functional forms and their functional invariants.* The theory of quadratic differential functional forms is of fundamental importance in a theory of Riemannian function space manifolds.

DEFINITION. *A functional expression*

$$g_{a\beta}[y]\,\delta y^a\,\delta y^\beta + g_a[y]\,(\delta y^a)^2;\quad g_{a\beta}[y] = g_{\beta a}[y] \tag{10.1}$$

will be said to be an absolute quadratic differential functional form if

$$(10.2)\qquad \bar{g}_{\alpha\beta}[\bar{y}]\,\delta\bar{y}^{\alpha}\,\delta\bar{y}^{\beta}+\bar{g}_{\alpha}[\bar{y}]\,(\delta\bar{y}^{\alpha})^{2}=g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+g_{\alpha}[y]\,(\delta y^{\alpha})^{2}$$

holds under an arbitrary functional transformation $\mathfrak{G}_s$.

We shall restrict ourselves in this paper to quadratic forms

$$(10.3)\qquad g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+\int_a^b(\delta y^{\alpha})^{2}\,d\alpha;\quad g_{\alpha\beta}[y]=g_{\beta\alpha}[y]$$

in which $g_{\alpha\beta}$ is a continuous functional of y^t and a continuous function of the variables α and β. A necessary and sufficient condition that

$$(10.4)\qquad \bar{g}_{\alpha\beta}[\bar{y}]\,\delta\bar{y}^{\alpha}\,\delta\bar{y}^{\beta}+\int_a^b(\delta\bar{y}^{\alpha})^{2}\,d\alpha=g_{\alpha\beta}[y]\,\delta y^{\alpha}\,\delta y^{\beta}+\int_a^b(\delta y^{\alpha})^{2}\,d\alpha$$

under an arbitrary transformation $\mathfrak{G}_s$ is that

$$(10.5)\qquad \begin{aligned}\bar{g}_{\alpha\beta}[\bar{y}]=g_{\alpha\beta}[y]&+g_{\sigma\beta}{}'y_{\alpha}{}^{\sigma}[\bar{y}]+g_{\alpha\sigma}{}'y_{\beta}{}^{\sigma}+g_{\sigma\tau}{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\tau}\\&+{}'y_{\beta}{}^{\alpha}+{}'y_{\alpha}{}^{\beta}+\int_a^b{}'y_{\alpha}{}^{\sigma}\,{}'y_{\beta}{}^{\sigma}\,d\sigma.\end{aligned}$$

On the basis of theorem (2. II) one can easily prove the following theorem.

THEOREM (10. I). *The Fredholm determinant* $D[g_{\alpha\beta}]$ *of the functional coefficients* $g_{\alpha\beta}[y]$ *of the form* (10. 3) *is a relative scalar invariant of weight two;* i. e.,

$$(10.6)\qquad D[\bar{g}_{\alpha\beta}]=(D[{}'y_{\mu}{}^{\lambda}])^{2}\,D[g_{\alpha\beta}]$$

We shall assume throughout our whole discussion that the Fredholm determinant

$$(10.7)\qquad D[g_{\alpha\beta}]\neq 0.$$

We observe that the Fredholm determinant $D[g_{\alpha\beta}]$ is a continuous functional of y^t. Hence if it does not vanish at a point $y^t=q^t$, it will continue not to vanish in the functional neighborhood of the point q^t.

THEOREM (10. II). *The resolvent kernel* $g^{\alpha\beta}[g_{\lambda\mu}]$ *of the functional coefficients of the form* (10. 3) *is an absolute functional invariant of the quadratic differential functional form* (10. 3) *while the first minor* $D^{\alpha\beta}[g_{\lambda\mu}]$ *of the Fredholm determinant* $D[g_{\lambda\mu}]$ *is a relative functional invariant of weight two.* The *law of transformation of* $g^{\alpha\beta}$ *and* $D^{\alpha\beta}$ *is given respectively by*

$$(10.8)\qquad \begin{aligned}\bar{g}^{\alpha\beta}[\bar{g}_{\lambda\mu}]=g^{\alpha\beta}[g_{\lambda\mu}]&+g^{\sigma\beta}\,{}'\bar{y}_{\sigma}{}^{\alpha}[y]+g^{\alpha\sigma}\,{}'\bar{y}_{\sigma}{}^{\beta}+g^{\sigma\tau}\,{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\tau}{}^{\beta}\\&+{}'\bar{y}_{\beta}{}^{\alpha}+{}'\bar{y}_{\alpha}{}^{\beta}+\int_a^b{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\sigma}{}^{\beta}\,d\sigma,\end{aligned}$$

$$(10.9)\qquad \begin{aligned}\bar{D}^{\alpha\beta}[\bar{g}_{\lambda\mu}]=(D[{}'y_{j}{}^{i}])^{2}\,\Big[&D^{\alpha\beta}[g_{\lambda\mu}]+D^{\sigma\beta}\,{}'\bar{y}_{\sigma}{}^{\alpha}+D^{\alpha\sigma}\,{}'\bar{y}_{\sigma}{}^{\beta}\\&+D^{\sigma\tau}\,{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\tau}{}^{\beta}-D[g_{\alpha\beta}]\,({}'\bar{y}_{\beta}{}^{\alpha}+{}'\bar{y}_{\alpha}{}^{\beta}+\int_a^b{}'\bar{y}_{\sigma}{}^{\alpha}\,{}'\bar{y}_{\sigma}{}^{\beta}\,d\sigma)\Big].\end{aligned}$$

In order to study the functional differential form (10. 3) effectively, we shall assume that the functional $g_{\alpha\beta}[y]$ has differentials of all orders of the Volterra form. We shall make the further desirable restriction that all the functional derivatives involved in the differentials are continuous functionals of y^ι as well as continuous functions of the variables on which they depend in particular.

The analogue of the *Christoffel symbols* of the first kind is the functional $[\alpha\beta, \gamma]$ that is defined by

$$[\alpha\beta, \gamma] = \tfrac{1}{2}({}'g_{\gamma\alpha,\beta} + {}'g_{\beta\gamma,\alpha} - {}'g_{\alpha\beta,\gamma}) \tag{10. 10}$$

in which ${}'g_{\alpha\beta,\gamma}[y]$ is the functional derivative of the coefficients $g_{\alpha\beta}[y]$ of the form (10. 3). It is a consequence of (10. 5) that the law of transformation of the functional $[\alpha\beta, \gamma]$ is

$$\begin{aligned}\overline{[\alpha\beta, \gamma]} = {} & [\alpha\beta, \gamma] + [\lambda\beta, \gamma]\, {}'y_\alpha{}^\lambda[\bar{y}] + [\alpha\lambda, \gamma]\, {}'y_\beta{}^\lambda + [\alpha\beta, \lambda]\, {}'y_\gamma{}^\lambda \\ & + [\alpha\lambda, \mu]\, {}'y_\alpha{}^\lambda\, {}'y_\beta{}^\mu + [\lambda\beta, \mu]\, {}'y_\alpha{}^\lambda\, {}'y_\gamma{}^\mu + [\lambda\mu, \gamma]\, {}'y_\alpha{}^\lambda\, {}'y_\beta{}^\mu \\ & + [\lambda\mu, \nu]\, {}'y_\alpha{}^\lambda\, {}'y_\beta{}^\mu\, {}'y_\gamma{}^\nu + g_{\lambda\gamma}\, {}''y^\lambda{}_{\alpha\beta} + g_{\lambda\mu}\, {}'y_\gamma{}^\lambda\, {}''y^\mu{}_{\alpha\beta} \\ & + {}''y^\gamma{}_{\alpha\beta} + \int_a^b {}'y_\gamma{}^\lambda\, {}''y^\lambda{}_{\alpha\beta}\, d\lambda.\end{aligned} \tag{10. 11}$$

In (10. 11) we understand, of course, that

$$\overline{[\alpha\beta, \gamma]} = \tfrac{1}{2}({}'\bar{g}_{\gamma\alpha,\beta}[\bar{y}] + {}'\bar{g}_{\beta\gamma,\alpha}[\bar{y}] - {}'\bar{g}_{\alpha\beta,\gamma}[\bar{y}]).$$

The functional $\Gamma^\iota{}_{\alpha\beta}[y]$ defined by

$$\Gamma^\iota{}_{\alpha\beta} = [\alpha\beta, i] + g^{\iota\sigma}[g_{\lambda\mu}]\, [\alpha\beta, \sigma] \tag{10. 12}$$

is an affine connection uniquely determined by the quadratic functional differential form (10. 3). The notation $\Gamma^\iota{}_{\alpha\beta}$ need not cause any confusion, since throughout this paragraph and the next we shall be concerned exclusively with affinely connected manifolds whose affine connection is uniquely determined by a quadratic form (10. 3).

Since $g^{\alpha\beta}$ is the resolvent kernel of $g_{\alpha\beta}[y]$, it follows readily from the theory of the Fredholm integral equation that the *tensor* functional identity

$$g_{\alpha\beta} + g^{\alpha\beta} + g_{\alpha\sigma}g^{\sigma\beta} = 0 \tag{10. 13}$$

holds for all continuous functions y^ι which are in a functional neighborhood of a point for which the Fredholm determinant $D[g_{\alpha\beta}]$ does not vanish. By means of (10. 8), (10. 11), (10. 12), (10. 13), (3. 5) and (3. 6) we obtain by calculation the law of transformation

$$\bar{\Gamma}^{i}{}_{\alpha\beta} + \bar{\Gamma}^{\sigma}{}_{\alpha\beta}{}'\bar{y}_{\sigma}{}^{i}[\bar{y}] = {}''y^{i}{}_{\alpha\beta}[\bar{y}] + \Gamma^{i}{}_{\alpha\beta} + \Gamma^{i}{}_{\sigma\beta}{}'y_{\alpha}{}^{\sigma} + \Gamma^{i}{}_{\alpha\sigma}{}'y_{\beta}{}^{\sigma} + \Gamma^{i}{}_{\sigma\tau}{}'y_{\alpha}{}^{\sigma}{}'y_{\beta}{}^{\tau} \tag{10.14}$$

in which

$$\bar{\Gamma}^{i}{}_{\alpha\beta} = \overline{[\alpha\beta, i]} + \bar{g}^{i\sigma}[\bar{y}]\ \overline{[\alpha\beta, \sigma]}.$$

This then shows the truth of our assertion about the functional $\Gamma^{i}{}_{\alpha\beta}$ as defined by (10.12).

The identity (10.13) enables us to re-write the defining relation (10.12) in the form

$$[\alpha\beta, i] = g_{i\lambda}\,\Gamma^{\lambda}{}_{\alpha\beta} + \Gamma^{i}{}_{\alpha\beta}. \tag{10.15}$$

We have thus expressed the Christoffel symbol $[\alpha\beta, i]$ of the first kind in terms of the Christoffel symbol $\Gamma^{i}{}_{\alpha\beta}$ of the second kind.

It is perhaps worth while to mention explicitly a few properties of the functional $g^{\alpha\beta}$ considered as a functional of y^{i}. The differential of $g^{\alpha\beta}$ is of the Volterra form

$$\delta g^{\alpha\beta}[y] = {}'g_{\gamma}{}^{\alpha\beta}\,\delta y^{\gamma}. \tag{10.16}$$

This can be seen most easily as follows. By definition we have

$$g^{\alpha\beta} = -\frac{D^{\alpha\beta}[g_{\lambda\mu}]}{D[g_{\lambda\mu}]} \tag{10.17}$$

in which $D^{\alpha\beta}[g_{\lambda\mu}]$ is the first minor of the Fredholm determinant $D[g_{\lambda\mu}]$. It is known * that the differentials of the functionals $D^{\alpha\beta}$ and D, considered as functionals of $g_{\lambda\mu}$, are linear homogeneous functionals of the differentials $\delta g_{\lambda\mu}$. But $g_{\lambda\mu}$ has a differential of the Volterra form. Hence the differential of $g^{\alpha\beta}$ when we consider $g^{\alpha\beta}$ as a functional of y^{i} takes the form (10.16).

To develop the functional invariant theory of the quadratic form (10.3) effectively we must impose further restrictions on the functional $g_{\alpha\beta}[y]$. Let us assume that $g_{\alpha\beta}$ has not only differentials of all orders but that it is given by an absolutely and uniformly convergent Volterra development

$$g_{\alpha\beta}[q^{i} + y^{i}] = g_{\alpha\beta}[q^{i}] + {}'g_{\alpha\beta,\gamma}[q^{i}]\ y^{\gamma} + (1/2\,!)\,{}''g_{\alpha\beta,\gamma\delta}[q^{i}]\ y^{\gamma}y^{\delta} + (1/3\,!)\,{}'''g_{\alpha\beta,\gamma\delta\epsilon}[q^{i}]\ y^{\gamma}y^{\delta}y^{\epsilon} + \cdots \tag{10.18}$$

about an arbitrarily given point q^{i} in our function space.

Since the functional $\Gamma^{i}{}_{\alpha\beta}[y]$ possesses functional derivatives and differentials of all orders, we can construct the curvature tensor $B^{i}{}_{\alpha\beta\gamma}$ and its successive covariant functional derivatives on the basis of the particular affine connection (10.12). The functional $B_{\alpha\beta\gamma\delta}[y]$ defined by

$$B_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}\,B^{\sigma}{}_{\beta\gamma\delta} + B^{\alpha}{}_{\beta\gamma\delta} \tag{10.19}$$

* Cf. G. C. Evans, *American Mathematical Monthly*, Vol. 34 (1927), p. 148. See also Fredholm, *loc. cit.*

will be called the covariant form of the curvature tensor $B^{i}{}_{\alpha\beta\gamma}$. One can show by a direct calculation that $B_{\alpha\beta\gamma\delta}$ is a covariant tensor of order four.

By means of (10.13) we can re-write formula (10.19) in the form

$$B^{\alpha}{}_{\beta\gamma\delta} = g^{\alpha\sigma} B_{\sigma\beta\gamma\delta} + B_{\alpha\beta\gamma\delta}. \tag{10.20}$$

Relations (10.19) and (10.20) are special cases of a general result. In fact, if $T^{\,i\ldots j\ldots k}_{a\ldots b\ldots c}$ is a functional tensor covariant of order l and contravariant of order m, then

$$g_{j\sigma} T^{\,i\ldots\sigma\ldots k}_{a\ldots b\ldots c} + T^{\,i\ldots j\ldots k}_{a\ldots b\ldots c} \tag{10.21}$$

is a functional tensor covariant of order $l+1$ and contravariant of order $m-1$ while

$$g^{b\sigma} T^{\,i\ldots j\ldots k}_{a\ldots\sigma\ldots c} + T^{\,i\ldots j\ldots k}_{a\ldots b\ldots c} \tag{10.22}$$

is a functional tensor covariant of order $l-1$ and contravariant of order $m+1$.

On the basis of the results of the preceding paragraphs we can obviously set up the machinery of covariant functional differentiation for functional tensors as well as for functionals with a law of transformation (10.5) or (10.8). Differentiating (10.5) functionally and eliminating the functional derivatives $''y^{i}{}_{\alpha\beta}$ by means of (10.14) we obtain the tensor law of transformation of the covariant derivative $g_{\alpha\beta;\gamma}$ of $g_{\alpha\beta}$. The explicit form of $g_{\alpha\beta;\gamma}$ is

$$g_{\alpha;\beta\gamma}[y] = {}'g_{\alpha\beta,\gamma} - g_{\lambda\beta}\Gamma^{\lambda}{}_{\alpha\gamma} - g_{\alpha\lambda}\Gamma^{\lambda}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\beta}{}_{\alpha\gamma}. \tag{10.23}$$

In a similar manner we obtain the covariant functional derivative $g_{;\gamma}{}^{\alpha\beta}$ of $g^{\alpha\beta}$

$$g_{;\gamma}{}^{\alpha\beta} = {}'g_{\gamma}{}^{\alpha\beta} + g^{\alpha\lambda}\Gamma^{\beta}{}_{\lambda\gamma} + g^{\lambda\beta}\Gamma^{\alpha}{}_{\lambda\gamma} + \Gamma^{\beta}{}_{\alpha\gamma} + \Gamma^{\alpha}{}_{\beta\gamma}. \tag{10.24}$$

On employing the defining relation (10.12) and the identity (10.13) in (10.23) we obtain the result

$$g_{\alpha\beta;\gamma} = 0. \tag{10.25}$$

We also have

$$g_{;\gamma}{}^{\alpha\beta} = 0. \tag{10.26}$$

To prove this, let us differentiate (10.13) functionally. We evidently obtain

$${}'g_{\alpha,\beta\gamma} + {}'g_{\gamma}{}^{\alpha\beta} + {}'g_{\alpha\sigma,\gamma} g^{\sigma\beta} + g_{\alpha\sigma}\,{}'g_{\gamma}{}^{\sigma\beta} = 0,$$

a Fredholm integral equation with $g_{\alpha\sigma}$ as its kernel and ${}'g_{\gamma}{}^{\alpha\beta}$ as the unknown. The unique continuous solution of this equation is

$${}'g_{\gamma}{}^{\alpha\beta} = -{}'g_{\alpha\beta,\gamma} - {}'g_{\alpha\sigma,\gamma} g^{\sigma\beta} - {}'g_{\sigma\beta,\gamma} g^{\sigma\alpha} - g^{\sigma\alpha}\, g^{\tau\beta}\, {}'g_{\sigma\tau,\gamma}. \tag{10.27}$$

Since

(10.28) $$'g_{\alpha\beta,\gamma}[y] = [\alpha\gamma, \beta] + [\beta\gamma, \alpha],$$

we find that

(10.29) $$'g_{\gamma}{}^{\alpha\beta} = -g^{\alpha\lambda}\Gamma^{\beta}{}_{\lambda\gamma} - g^{\lambda\beta}\Gamma^{\alpha}{}_{\lambda\gamma} - \Gamma^{\beta}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\gamma}.$$

Substituting in (10.24) this expression for $'g_{\gamma}{}^{\alpha\beta}$ we finally obtain (10.26).

It is a consequence of (10.25) that the covariant derivatives $B_{\alpha\beta\gamma\delta;\epsilon}$ of $B_{\alpha\beta\gamma\delta}$ is given by

(10.30) $$B_{\alpha\beta\gamma\delta;\epsilon} = g_{\alpha\sigma}B^{\sigma}{}_{\beta\gamma\delta;\epsilon} + B^{\alpha}{}_{\beta\gamma\delta;\epsilon}.$$

More generally we have

(10.31) $$B_{\alpha\beta\gamma\delta;\,\epsilon;\,\zeta;\,\ldots;\,\kappa} = g_{\alpha\sigma}B^{\sigma}_{\beta\gamma\delta;\,\epsilon;\,\zeta;\,\ldots;\,\kappa} + B^{\alpha}_{\beta\gamma\delta;\,\epsilon;\,\zeta;\,\ldots;\,\kappa}$$

Let $\eta_{\alpha\beta}[z]$ denote the coefficients $g_{\alpha\beta}[y]$ in a normal coördinate system z^{i} with origin at the point $y^{i} = q^{i}$ and based on the affine connection (10.12). On account of (10.25), the Volterra development for $\eta_{\alpha\beta}[z]$ takes the form

(10.32) $$\eta_{\alpha\beta}[z] = \eta_{\alpha\beta}[q] + (1/2!)g_{\alpha\beta;\gamma\delta}[q]\,z^{\gamma}z^{\delta} + (1/3!)g_{\alpha\beta;\gamma\delta\epsilon}z^{\gamma}z^{\delta}z^{\epsilon} + \cdots$$

where the tensor $g\,{}_{\alpha\beta;\,\gamma_1\ldots\gamma_p}[y]$ is the pth extension of the functional $g_{\alpha\beta}[y]$.

The importance of this sequence of functional tensors is made evident by the following theorem.

THEOREM (10.III). *Any tensor functional invariant of order r of the quadratic functional form* (10.3)

(10.33) $$T^{ij\ldots l}_{ab\ldots c}[g_{\alpha\beta}, 'g_{\alpha\beta,\gamma_1}, \cdots, {}^{(r)}g_{\alpha\beta,\gamma_1\ldots\gamma_r}]$$

can be put into the form

(10.34) $$T^{ij\ldots l}_{ab\ldots c}[g_{\alpha\beta}, 0, g_{\alpha\beta,\gamma_1\gamma_2}, \cdots, g_{\alpha\beta,\gamma_1\ldots\gamma_r}]$$

by replacing $g_{\alpha\beta}$ by $g_{\alpha\beta}$, $'g_{\alpha\beta,\gamma_1}$ by zero, and the higher functional derivatives of $g_{\alpha\beta}$ by the corresponding extensions of $g_{\alpha\beta}$. The proof of this theorem is entirely similar to that of Theorem (9.I). An obvious modification of Theorem (10.III) enables us to give a replacement theorem for the analogues of the n-dimensional differential parameters.

Let $f[y]$ be a scalar functional of y^{i} such that

(10.35) $$\delta f = 'f_{\alpha}\,\delta y^{\alpha}, \quad \delta' f_{\alpha} = ''f_{\alpha\beta}\,\delta y^{\beta}$$

The functional

(10.36) $$g^{\alpha\beta}[y]\,'f_{\alpha;\beta}[y] + \int_{a}^{b} 'f_{\alpha;\alpha}[y]\,d\alpha \quad ('f_{\alpha;\beta} = ''f_{\alpha\beta} - 'f_{\sigma}\Gamma^{\sigma}{}_{\alpha\beta} = f_{;\alpha\beta})$$

is a tensor and the functional equation with functional derivatives

$$g^{\alpha\beta}\,{}'f_{\alpha;\beta} + \int_a^b {}'f_{\alpha;\alpha}[y]\,d\alpha = 0 \tag{10.37}$$

is an analogue of Laplace's equation.

The functional

$$g^{\alpha\beta}\,{}'f_\alpha\,{}'f_\beta + \int_a^b (f_\alpha)^2\,d\alpha$$

and

$$g^{\alpha\beta}\,{}'f_\alpha\,{}'\phi_\beta + \int_a^b {}'f_\alpha\,{}'\phi_\alpha\,d\alpha$$

are analogues of the two n-dimensional differential parameters of the first order.

11. *Riemannian function space manifolds.* A quadratic functional differential form evaluated at a fixed point y^t is an algebraic quadratic functional form in the differentials δy^t. With this in mind let us make some further observations on algebraic quadratic functional forms

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\,d\alpha. \tag{11.1}$$

Suppose that (11.1) is a *positive definite functional form.* Analytically this means that

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2\,d\alpha > 0 \tag{11.2}$$

for all functions y^α which do not vanish identically on the interval (a, b). It is easily shown that the *Fredholm determinant* $D[g_{\alpha\beta}]$ *of a positive definite form* (11.1) *does not vanish* (i. e., unity is not a characteristic value of $D[g_{\alpha\beta}]$). In fact suppose

$$D[g_{\alpha\beta}] = 0.$$

Then the homogeneous Fredholm integral equation

$$y^\alpha + g_{\alpha\beta}y^\beta = 0$$

will possess a continuous solution y^α which does not vanish identically. Obviously this contradicts (11.2).

Clearly the group of Fredholm transformations leaves invariant the positive definite property of a form (11.1).

Let us consider now a positive definite quadratic differential form

$$g_{\alpha\beta}[y]\,\delta y^\alpha\,\delta y^\beta + \int_a^b (\delta y^\alpha)^2\,d\alpha. \tag{11.3}$$

We shall understand that

$$g_{\alpha\beta}\,\delta y^\alpha\,\delta y^\beta + \int_a^b (\delta y^\alpha)^2\,d\alpha > 0 \tag{11.4}$$

for all continuous functions y^i such that

$$| q^i - y^i | < N$$

and for all differentials δy^i which do not vanish identically on the interval (a, b).

A function space manifold possessing a positive definite quadratic form (11. 3) will be called a *Riemannian function space manifold.* We shall define the element of length ds by

$$ds^2 = g_{\alpha\beta}[y]\, \delta y^\alpha\, \delta y^\beta + \int_a^b (\delta y^\alpha)^2\, d\alpha = Q[\delta y, \delta y]. \tag{11. 5}$$

Let

$$y^i = y^i(u)$$

be the equation of a parameterized curve with a real parameterization u. The *length* s of this curve between the points $y^i(u_0)$ and $y^i(u_1)$ will by definition be

$$s = \int_{u_0}^{u_1} \left[g_{\alpha\beta}[y^i(u)]\, \frac{\partial y^\alpha(u)}{\partial u}\, \frac{\partial y^\beta(u)}{\partial u} + \int_a^b \left(\frac{\partial y^\alpha(u)}{\partial u} \right)^2 d\alpha \right]^{1/2} du. \tag{11. 6}$$

The functional s so defined is a function of function space curves since its value depends only on the curve and not on any accidental parameterization u or coördinate system y^i.

The metrical ground-form (11. 5) may be used to define the angle θ between two direction vectors $\delta_1 y^\alpha$ and $\delta_2 y^\alpha$. By definition

$$\cos\theta = Q[\delta_1 y, \delta_2 y] / \left(Q[\delta_1 y, \delta_1 y] \right)^{1/2} \left(Q[\delta_2 y, \delta_2 y] \right)^{1/2}. \tag{11. 7}$$

Since the metrical ground-form $Q[\delta y, \delta y]$ is a positive definite functional form, it follows that

$$\left(Q[\delta_1 y, \delta_2 y] \right)^2 < Q[\delta_1 y, \delta_1 y]\; Q[\delta_2 y, \delta_2 y] \tag{11. 8}$$

for all non-proportional functions $\delta_1 y^i$ and $\delta_2 y^i$.

If the parameter u is taken to be the arc length s, we evidently have

$$g_{\alpha\beta}\, [y^i(s)]\, \frac{\partial y^\alpha(s)}{\partial s}\, \frac{\partial y^\beta(s)}{\partial s} + \int_a^b \left(\frac{\partial y^\alpha(s)}{\partial s} \right)^2 d\alpha = 1. \tag{11. 9}$$

The problem of finding the *geodesics* of our Riemannian function space manifold is a problem in *minima of functionals.* The functional to be minimized is the distance functional (11. 6). The generalized Euler-Lagrange equations for this variational problem are readily found to be

$$\frac{\partial^2 y^\alpha(s)}{\partial s^2} + g_{\alpha\beta}[y^i(s)]\, \frac{\partial^2 y^\beta(s)}{\partial s^2} + [\gamma\delta, \alpha]\, \frac{\partial y^\gamma(s)}{\partial s}\, \frac{\partial y^\delta(s)}{\partial s} = 0 \tag{11. 10}$$

in which

$$[\gamma\delta, \alpha] = \tfrac{1}{2}\left({}'g_{\alpha\gamma,\delta}[y^i(s)] + {}'g_{\delta\alpha,\gamma}[y^i(s)] - {}'g_{\gamma\delta,\alpha}[y^i(s)]\right).$$

We may look upon (11.10) as a Fredholm integral equation to be solved for $\partial^2 y^\alpha(s)/\partial s^2$. The unique solution, continuous as a function of α, is given by

$$\frac{\partial^2 y^\alpha(s)}{\partial s^2} + \Gamma^\alpha{}_{\gamma\delta}[y^i(s)]\frac{\partial y^\gamma(s)}{\partial s}\frac{\partial y^\delta(s)}{\partial s} = 0. \tag{11.11}$$

In (11.11) the functional $\Gamma^\alpha{}_{\gamma\delta}[y^i(s)]$ is the affine connection (10.12) evaluated for the functional argument $y^i(s)$. The solution (11.11) is, of course, valid for the functional neighborhood in which unity is not a characteristic value of the kernel $g_{\alpha\beta}[y^i(s)]$.

The integro-differential equation (11.11) is a special case of the equation (6.9). It is to be observed that in choosing the initial conditions

$$\begin{aligned} q^i &= y^i(s_0) \\ p^i &= (\partial y^i(s)/\partial s)_{s=s_0} \end{aligned} \tag{11.12}$$

for the equation (11.11), the function p^i must be taken so as to satisfy the condition

$$g_{\alpha\beta}[q^i]\, p^\alpha p^\beta + \int_a^b (p^\alpha)^2\, d\alpha = 1. \tag{11.13}$$

In case our Riemannian manifold is a Euclidean function space, for which

$$ds = \left[\int_a^b [\partial y^\alpha(u)/\partial u]^2\, d\alpha\right]^{\frac{1}{2}} du, \tag{11.14}$$

the equations of the geodesics, parameterized with the arc length s, take the simple form

$$\partial^2 y^\alpha(s)/\partial s^2 = 0. \tag{11.15}$$

The most general solution of (11.15) obviously is

$$y^\alpha = A^\alpha s + B^\alpha.$$

The methods of this paper can be extended to function spaces consisting of continuous functions in two or more independent variables. One might suspect that Volterra's theory of *permutable functions* and *functions of composition* would be useful in a geometry of functions of two independent variables.

THE OHIO STATE UNIVERSITY.

A Theorem Concerning the Affine Connection.

By Tracy Yerkes Thomas.

Consider the equations of the paths:

$$d^2x^i + \Gamma^i_{\alpha\beta} dx^\alpha dx^\beta = 0 \tag{1}$$

of an affinely connected space of affine connection $\Gamma^i_{\alpha\beta}$ and the equations

$$d^m x^i + \Gamma^i_{\alpha\beta\ldots\epsilon} dx^\alpha dx^\beta \cdots dx^\epsilon = 0 \qquad (m = 3, 4, \cdots,) \tag{2}$$

which result from them by differentiation.

The affine connection $\Gamma^i_{\alpha\beta}$ transforms into the connection $C^i_{\alpha\beta}$ according to the equations

$$C^\sigma_{ab} \frac{\partial x^i}{\partial y^\sigma} = \frac{\partial^2 x^i}{\partial y^a \partial y^b} + \Gamma^i_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \tag{3}$$

when the coördinates x^i undergo an arbitrary analytic transformation by which they are transformed into a set of coördinates y^i. In order to see how the generalized connections $\Gamma^i_{\alpha\beta\ldots\epsilon}$ transform under this same transformation of coördinates let us examine how the equations (3) themselves are obtained. Differentiating the equations of transformation of the differentials dx^i, namely

$$dx^i = \frac{\partial x^i}{\partial y^a} dy^a \tag{4}$$

we have

$$d^2x^i = \frac{\partial^2 x^i}{\partial y^a \partial y^b} dy^a dy^b + \frac{\partial x^i}{\partial y^a} d^2y^a. \tag{5}$$

Substituting (4) and (5) into the equations (1) we find that the condition that (1) should remain invariant in form is that the $\Gamma^i_{\alpha\beta}$ should transform by (3). Suppose now that it is possible to continue in this way and so get the equations of transformation of the first $p-1$ successive connections $\Gamma^i_{\alpha\beta\ldots\mu}$ as the result of the condition that the equations (1) and (2) for $m = 3, \cdots, p-1$ should be invariant in form. In the y^i coördinates these equations then become

$$d^m y^i + C^i_{\alpha\beta\ldots\mu} dy^\alpha dy^\beta \cdots dy^\mu = 0 \qquad (m = 2, \cdots, p-1). \tag{6}$$

To find the equations of transformation of the connection $\Gamma^i_{\alpha\beta\ldots\epsilon}$ in (2) for $m = p$ we differentiate (5) repeatedly so as to obtain

$$(7) \qquad d^p x^i = \frac{\partial^p x^i}{\partial y^a \dots \partial y^t} \, dy^a \cdots dy^t + \frac{\partial x^i}{\partial y^\sigma} \, d^p y^\sigma + \bigstar$$

where the $\bigstar$ indicates terms involving differentials $d^m x^i$ for which $m < p$. These differentials are to be eliminated by means of (6) after which (7) is to be substituted into (2) for $m = p$. The requirement of the invariance of form then gives

$$(8) \qquad C^\sigma{}_{a \dots t} \frac{\partial x^i}{\partial y^\sigma} = \frac{\partial^p x^i}{\partial y^a \cdots \partial y^t} + \Gamma^i{}_{a \dots \epsilon} \frac{\partial x^a}{\partial y^a} \cdots \frac{\partial x^\epsilon}{\partial y^t} + \bigstar$$

where the $\bigstar$ denotes a polynominal composed of a sum of terms each of which involves a component of connection $C^i{}_{a \dots \mu}$ with less than p subscripts. The equations of transformation of an arbitrary connection $\Gamma^i{}_{a\beta \dots \epsilon}$ are thus given by (8).

Now consider a set of functions $Z^i(x)$ defined by the power series expansion

$$(9) \qquad Z^i(x) = \sum_{s=2}^{\infty} (1/s!) (\Gamma^i{}_{\sigma_1 \dots \sigma_s})_q (x^{\sigma_1} - q^{\sigma_1}) \cdots (x^{\sigma_s} - q^{\sigma_s})$$

in which the constants q^i are arbitrary and the coefficients are subject only to the condition that the series converges. The following theorem regarding the series (9) in its relation to the affine connection $\Gamma^i{}_{a\beta}(x)$ can be proved.

THEOREM. *Given an arbitrary affine connection and a set of arbitrary functions $Z^i(x)$ defined by* (9). *Then there exists a coördinate system x such that the components of the affine connection $\Gamma^i{}_{a\beta}(x)$ and of the generalized connections $\Gamma^i{}_{a\beta \dots \epsilon}(x)$ at $x^i = q^i$ will take on the values of the coefficients of the power series expansion of the function $Z^i(x)$.*

To prove this theorem we transform the components of the given affine connection to a system of normal coördinates y^i. In the normal coördinates y^i let the components of affine connection be represented by the set of functions $C^i{}_{a\beta}(y)$. Now construct the transformation

$$(10) \qquad x^i = q^i + yi - \sum_{s=2}^{\infty} (1/s!) (\Gamma^i{}_{\sigma_1 \dots \sigma_s})_q y^{\sigma_1} \cdots y^{\sigma_s}$$

which is determined by the coefficients of the expansion of the functions Z^i. All components $C^i{}_{a\beta}(y)$ vanish at $y^i = 0$ since the coördinates y^i are normal coördinates. Hence if we evaluate both members of (8) at the origin of the coördinates y^i we have

$$\Gamma^i{}_{a \dots \epsilon}(q) = - \left(\frac{\partial^p x^i}{\partial y^a \cdots \partial y^\epsilon} \right)_0$$

which means on account of (10) that at $x^i = q^i$ the components of the connection $\Gamma^i{}_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^i{}_{\alpha\beta\ldots\epsilon}(x)$ are equal to the coefficients of the power series (9).

In particular the affine connection of the above theorem may be a Christoffel symbol:

$$\tfrac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)$$

based on a fundamental metric tensor $g_{\alpha\beta}$.

If we take the given connection in the above theorem to be that of a flat space (a remark by Professor Veblen) we have the

SPECIAL THEOREM. *For a flat space (affine or Riemann) there exists a coördinate system x such that the components of the affine connection $\Gamma^i{}_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^i{}_{\alpha\beta\ldots\epsilon}(x)$ at $x^i = q^i$ will take on the values of the coefficients of the power series expansion of the functions $Z^i(x)$.*

In certain investigations it is of importance to consider invariantive systems of partial differential equations which involve the quantities $g_{\alpha\beta}$ or $\Gamma^i{}_{\alpha\beta}$ as unknown functions. A simple system of equations of this sort is the system of Einstein's gravitational equations for free space which expresses the vanishing of the contracted curvature tensor. The above functions $Z^i(x)$ belong to the set of arbitrary constants and functions which are determinative of the solutions of such differential equations.

PRINCETON UNIVERSITY.

Concerning Subsets of a Continuous Curve which can be Connected through the Complement of the Continuous Curve.*

By W. L. Ayres.

1. *Introduction.* One of the present tendencies in the theory of sets of points is the replacing of theorems concerning subsets of space by theorems concerning subsets of a continuous curve. This is evidently a more general point of view since the entire space is a particular case of a continuous curve.† But in making this change it is found that certain theorems involving the notion of connectedness no longer hold true. In studying these difficulties we are led to the introduction of a new notion of a subset of a point set being connected through the complement of the point set.‡ In this paper we make a study of this new property and, among other results, show that a number of theorems remain true on replacing the entire space by a continuous curve if the property of connectedness is replaced by this new property.

It is understood that all the point sets which are considered in this paper lie in a two-dimensional Euclidean space, which we denote by S.

2. *Definitions.* If the point set N is a subset of the point set M and x and y are points of N, we shall say that x and y *can be connected through* N *and* $S - M$ if the set $N + (S - M)$ contains a connected subset containing both x and y. If the set $N + (S - M)$ contains a continuum containing both x and y, we shall say that x and y *can be connected through* N *and* $S - M$ *in the strong sense.* If every two points of N can be connected through N and $S - M$, we shall say that N is connected through N and $S - M$.

A plane domain D will be said to be *simply connected* if, for every simple

* Presented to the American Mathematical Society September 9 and October 29, 1927.

† We are using the term continuous curve in the so-called "generalized sense," i. e. a set which is closed, connected and connected im kleinen is called a **continuous** curve. This was suggested independently by R. L. Moore and by S. Mazurkiewicz. See Moore, "Concerning Simple Continuous Curves," *Transactions of the American Mathematical Society*, Vol. 21 (1920), footnote on page 347, and Mazurkiewicz, "Sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. I (1920), page 193.

‡ Independently of the author and apparently almost simultaneously, R. L. Moore noticed this new property. See his paper, "Some Separation Theorems, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 711-716.

closed curve J lying wholly in D, either the interior or the exterior of J is a subset of D.* A connected subset D of a continuous curve M is said to be an *M-domain* if $M - D$ is closed.† We shall say that an M-domain D is *simply connected with respect to* M if, for every simple closed curve J lying wholly in D, either D contains every point of M which lies in the interior of J or D contains every point of M which lies in the exterior of J.

Notations. We shall use the term *arc* as meaning *simple continuous arc.* If x and y are points of an arc α, the subarc xy of α is understood to mean the arc whose end-points are x and y and which is a subset of α. If xy denotes an arc with endpoints x and y, the symbols $<xy, xy>$ and $<xy>$ denote the sets $xy - x$, $xy - y$ and $xy - x - y$ respectively. If x and y are points, $d(x, y)$ denotes the distance from x to y. If x is a point and K is a point set, $d(x, K)$ denotes the lower bound of the numbers $d(x, y)$ for all points y of K. The symbol $[X]$ denotes a set each of whose elements may be represented by X. The elements X may be points or sets of points. If $N(P)$ is a point set defined relative to the point P, the symbol

$$\sum_{P \subset K} N(P)$$

denotes the set of all points Q such that Q belongs to some set $N(P)$ for some point P of the point set K.

3. THEOREM 1. *If the point set K is a subset of the point set M and P is a point of K, then the set of all points of K which can be connected to P through K and $S - M$ is closed with respect to K.*‡

Proof. Let K_p denote the set of all points of K which can be connected to P through K and $S - M$, and let X be any limit point of K_p which belongs to K. For each point Q of K_p there exists a connected subset C_Q of $K + (S - M)$ containing P and Q. Let

$$Z = \sum_{Q \subset K_p} C_Q.$$

* This definition differs slightly from that given by R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), p. 257. According to our definition the exterior of a circle is a simply connected domain, while by Moore's definition it is not simply connected. For bounded domains the two definitions are equivalent.

† R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), p. 341.

‡ A point set G is said to be closed with respect to the point set H if every limit point of G which belongs to H belongs also to G. Cf. Knaster and Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2 (1921), p. 207.

The set Z is connected since every set C_Q is connected and contains the point P. Also the set Z contains the set K_p. Since X is a limit point of the set K_p, it is a limit point of the set Z. Then $X + Z$ is a connected subset of $K + (S - M)$ containing X and P. By definition of K_p, the point X belongs to the set K_p. Then every limit point of K_p which belongs to K belongs also to K_p and thus K_p is closed with respect to K.

THEOREM 2. *In order that a bounded M-domain D of a continuous curve M be simply connected with respect to M it is necessary and sufficient that the M-boundary B of D be connected through B and $S - M$.*

Proof. The condition is necessary. If M is the entire plane then, by a theorem of R. L. Moore,* the M-boundary B is connected and thus is connected through B and $S - M$. If M is not the entire plane, let P be some point of $S - M$ and let T denote an inversion of the plane with respect to some circle with center at P. Let M' denote $T(M)$ or $T(M) + P$ according as M is or is not bounded. Then M' is a bounded continuous curve,† $T(D)$ is an M'-domain which is simply connected with respect to M' and has $T(B)$ as its M'-boundary. Now let us suppose that B is not connected through B and $S - M$. Since B is bounded and closed, $T(B)$ is bounded, closed, does not contain P and is not connected through $T(B)$ and $S - M'$. Then $T(B)$ contains two points x and y which cannot be connected through $T(B)$ and $S - M'$. By a theorem due to R. L. Moore,‡ there exists a simple closed curve J such that J belongs to M', separates x and y in S and has no point in common with $T(B)$. Since $T(B)$ contains points (x and y) in both the interior and exterior of J, the M'-domain $T(D)$ contains points in both the interior and exterior of J. Then, as $T(D)$ is connected, $T(D)$ contains a point of J. If there is a point of J which does not belong to $T(D)$, then J must contain at least one point of $T(B)$ contrary to the condition that $T(B)$ and J have no common points. Hence J lies wholly in $T(D)$. But there is a point of $M' - T(D)$ in both the interior and exterior of J and thus $T(D)$ is not simply connected with respect to M'. But this is a contradiction and therefore B must be connected through B and $S - M$.

The condition is sufficient. Suppose D is not simply connected with respect to M. Then there exists a simple closed curve J which lies wholly in D and such that $M - D$ contains a point x in the interior of J and a point

* "Concerning Continuous Curves in the Plane," *loc. cit.*

† Knaster and Kuratowski, "Sur les continus non-bornés," *Fundamenta Mathematicae*, Vol. 5 (1924), pp. 23-58.

‡ "Some Separation Theorems," *loc. cit.*, Theorem 3.

y in the exterior of J. Let α be an arc of M whose endpoints are x and y. In the order x to y let $\bar{x}$ be the first point of J on α and $\bar{y}$ be the last point. On the subset $x\bar{x}>$ of α there is a point x' of B and on the subset $y\bar{y}>$ of α there is a point y' of B. Since B is connected through B and $S-M$, there is a connected subset K of $B+(S-M)$ which contains the points x' and y'. The connected set K has no point in common with D and contains the points x' and y' in the interior and exterior of J. But as J is a subset of D, this is impossible.

THEOREM 3. *In order that a bounded and connected subset D of a continuous curve M be an M-domain which is simply connected with respect to M it is necessary and sufficient that there exist a simply connected domain R such that $R \cdot M = D$.*

Proof. Obviously the condition is sufficient. We shall show that it is also necessary. By Theorem 2, the M-boundary B of D is connected through B and $S-M$. There exists a continuum K such that $K \cdot M = B$.* For each point P of D there is a circle with center at P whose interior $I(P)$ contains no point of $K+(M-D)$. Let

$$G = \sum_{P \subset D} I(P).$$

The set G is a domain such that $G \cdot M = D$. Let J be any simple closed curve which lies wholly in G. Suppose that both the interior and exterior of J contain a point of $M-D$. Then it is easily seen that both the interior and exterior of J contain a point of B. The continuum K contains a point interior to J and a point exterior to J but contains no point of G. But this is impossible since J is a subset of G. Then either the interior or the exterior of J contains no point of $M-D$, and let $V(J)$ denote that complementary domain of J containing no point of $M-D$. Let R be the domain consisting of the domain G plus all the domains $V(J)$ for all simple closed curves J of G. Evidently $R \cdot M = D$. It remains to show that R is simply connected.

If R is not simply connected, there exists in R a simple closed curve J', such that the interior and exterior of J' contain points x and y of $S-R$. From the definition of R it is seen that there can exist no simple closed curve

* R. L. Moore, "Some Separation Theorems," *loc. cit.*, Theorem 4. Moore proves that if H is a closed subset of a continuous curve M and x and y are points of H which can be connected through H and $S-M$ then x and y can be connected through H and $S-M$ in the strong sense, but his proof is sufficient to show that if G is any subset of H which is connected through H and $S-M$ then G is connected through H and $S-M$ in the strong sense.

lying wholly in G and having x in its interior and y in its exterior. We shall obtain a contradiction by proving that such a simple closed curve does exist. If P is any point of J' which belongs to G, let $J_1(P)$ be a circle such that every point of $J_1(P)$ and its interior $L(P)$ belong to G. If P is any point of J' which belongs to $R - G$, by definition of R, there is a domain $L(P)$ belonging to R and containing P and whose boundary is a simple closed curve $J_1(P)$ belonging to G. Then J' is covered by the set of domains $[L(P)]$. By the Borel-Lebesgue theorem, there is a finite subset, $L(P_1), L(P_2), \cdots, L(P_n)$, of $[L(P)]$ which covers J'. Neither the point x nor the point y belongs to the set $\sum_{i=1}^{n}(L(P_i) + J_1(P_i))$ and it is not difficult to see that the the boundary of the complementary domain of $\sum_{i=1}^{n}(L(P_i) + J_1(P_i))$ containing x is a simple closed curve J'' which belongs to $\sum_{i=1}^{n} J_1(P_i)$ and thus to G. The interior of J'' is a subset of the interior of J' and thus y lies in the exterior of J''. We have the desired contradiction. Hence R must be simply connected.

THEOREM 4. *If M is a continuous curve and K is a bounded cutting* of M such that if H is any maximal connected subset of K the set K-H is not a cutting of M, then K is connected through K and S-M.*

Proof. Let A and B be points of $M - K$ which lie in no connected subset of $M - K$. By a theorem of G. T. Whyburn,† the set K contains a set G which is an irreducible cutting of M between A and B. Evidently every maximal connected subset of K contains a point of G. Let

$$M - G = M_1 + M_2,$$

where M_1 and M_2 are mutually separated sets containing A and B respectively. If the set G is not connected through G and S—M, there exists a simple closed curve J lying in M, separating two points x and y of G in S and having no point in common with G. Then J lies wholly in M_1 or wholly in M_2. Suppose it lies in M_1. By a theorem due to G. T. Whyburn,‡ the set $M_2 + G$

* A subset K of a continuum M is said to be a *cutting* of M if $M - K$ is not connected. The set K is said to be an *irreducible cutting of M between the points A and B* if $M - K$ is the sum of two mutually separated sets M_A and M_B containing A and B respectively and no subset of K has this property. Cf. G. T. Whyburn, "Concerning Irreducible Cuttings of a Continuum," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 408 (abstract).

† *Loc. cit.*

‡ *Loc. cit.*

is a continuum. But this is impossible since J and $M_2 + G$ have no common points and G contains a point in both the interior and exterior of J. Hence $S - M$ contains a set L such that $G + L$ is connected. Since every maximal connected subset of K contains a point of G, the set $K + L$ is connected.

THEOREM 5. *In order that a continuum M be a continuous curve it is necessary and sufficient that if A is an arc, K is a closed subset of $M + A$ and x and y are points of K which can be connected through K and $S - (M + A)$, then x and y can be connected through K and $S - (M + A)$ in the strong sense.*

R. L. Moore * has shown that the condition is necessary and the condition may be shown to be sufficient by a modification of the proof of another theorem of his.†

THEOREM 6. *Let M be a continuous curve and N be a closed subset of M. For each point P of N let $C(P)$ be the set of all points of S whose distance from P is less than or equal to a number $r(P)$. Let $K(P)$ be the maximal connected subset of $M \cdot C(P)$ which contains P. Suppose* (1) *the function $r(P)$ is a bounded non-negative upper semi-continuous function‡ of the point P over the set N,* (2) *the set of points $[P]$ for which $r(P) = 0$ contains no continuum. Under these conditions every maximal connected subset of the set*

$$\sum_{P \subset N} K(P)$$

is a continuous curve.

Proof. Let H be a maximal connected subset of $\sum_{P \subset N} K(P)$. Let A be any limit point of H. Let $P_1, P_2, P_3, \cdots$, be a sequence of points of H having A as a sequential limit point. Let P'_i denote the center of the circle $C(P)$ which contains P_i. Since the function $r(P)$ is bounded, there exists a sequence of integers $n_1, n_2, n_3, \cdots$, such that the sequence $P'_{n_1}, P'_{n_2}, P'_{n_3}, \cdots$, has a sequential limit point B. The set N being closed, the point B belongs

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* "Some Separation Theorems," *loc. cit.*, Theorem 4. Moore proves only the bounded case and makes use of the fact that only a finite number of the complementary domains of a continuous curve are of diameter greater than a given positive number. While this is not true for unbounded continuous curves, it is true, and this would be sufficient for Moore's proof, that only a finite number of the complementary domains of a continuous curve (bounded or not) that contain points within a given circle are of diameter greater than a given positive number.

† *Ibid.*, Theorem 5.

‡ E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge University Press, 2nd Edition 1921, Vol. I, p. 290.

to N. For every value of i, $r(P'_{n_i}) \geqq d(P_{n_i}, P'_{n_i})$. Then as $r(P)$ is upper semi-continuous, $r(B) \geqq d(A, B)$. The sets $K(P'_{n_1}), K(P'_{n_2}), K(P'_{n_3}), \cdots$, have a closed and connected limiting set containing both A and B.* Moreover this limiting set belongs to M and to $C(B)$, and thus it belongs to $K(B)$. Then A belongs to $K(B)$ and thus to H. Therefore H is a closed set.

If H is not a continuous curve, there exist two concentric circles C_1 and C_2 and an infinite sequence of subcontinua of H, $H_\omega, H_1, H_2, H_3, \cdots$, satisfying the conditions of the Moore-Wilder Lemma.† Let r_1 and r_2 be the radii of C_1 and C_2 and let $r = r_2 - r_1 > 0$. Let C_3 and C_4 be circles concentric with C_1 and with radii $r_1 + r/3$ and $r_2 - r/3$ respectively. Let $\bar{H}_\alpha$ ($\alpha = \omega, 1, 2, 3, \cdots$) be the set of all points of H_α which lie on C_3 or on C_4 or between C_3 and C_4. For each point x of H let x' denote the center of the corresponding circle $C(P)$ containing x and used in defining the set $\sum_{P \subset N} K(P)$.

In case the maximum value of $r(x')$ for all points x of $\bar{H}_n$ approaches zero as n increases indefinitely, every point of $\bar{H}_\omega$ is a limit point of N and thus N contains $\bar{H}_\omega$. By condition (2) of the hypothesis of the theorem, the set $\bar{H}_\omega$ must contain a point Q such that $r(Q) > 0$. Let η be the smaller of the numbers $r(Q)$ and $r/3$. There exists a positive number δ_η such that every point of M whose distance from Q is less than δ_η can be joined to Q by an arc of M whose diameter is less than η. There exists an integer $\bar{n}$ such that $H_{\bar{n}}$ contains a point whose distance from Q is less than δ_η. Then $K(Q)$ contains a point of $H_{\bar{n}}$. But this is contrary to the condition that every set H_n is a maximal connected subset of $H \cdot G$, where G denotes the set consisting of C_1, C_2 and all those points of S which lie between C_1 and C_2.

If the above condition does not hold, there exists a positive number ϵ and an infinite sequence of integers $n_1, n_2, n_3, \cdots$, such that $\bar{H}_{n_i}$ contains a point P_i so that (a) $r(P'_i) > \epsilon$, (b) $[P_i]$ has a sequential limit point U on $\bar{H}_\omega$, (c) $[P'_i]$ has a sequential limit point V. Let

$$\overline{\lim_{i \to \infty}}\, r(P'_i) = m \geqq \epsilon.$$

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* Z. Janiszewski, "Sur les continus irréductibles entre deux points," *Journal de l'Ecole Polytechnique*, Ser. 2, Vol. 16 (1912), p. 109.

† For a full statement of the lemma, see R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, p. 371. See also the earlier papers of R. L. Moore, "A Characterization of Jordan Regions by Properties that have no Reference to Their Boundaries," *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 364-370, and "Continuous Sets that have no Continuous Sets of Condensation," *Bulletin of the American Mathematical Society*, Vol. 25 (1919), pp. 174-176.

Since $r(P)$ is an upper semi-continuous function, $r(V) \geqq m$. The limiting set of $K(P'_1)$, $K(P'_2)$, $K(P'_3)$, $\cdots$, is closed, connected and contains U and V. If t denotes the smaller of the numbers $\frac{1}{2}\epsilon$ and $r/8$, then all but a finite number of the sets $K(P'_i)$ contain a point x_i whose distance from P_i is less than $r/8$ and lying interior to the circle C_5 with center at V and radius $r(V) - t/2$. The points $[x_i]$ have a limit point $\bar{x}$ which must belong to H_ω since $d(\bar{x}, U) \leqq r/8$, and $\bar{x}$ must lie interior to or on the circle C_5 and belong to $K(V)$. Since M is a continuous curve there exists a number δ_t such that every point of M whose distance from $\bar{x}$ is less than δ_t can be joined to $\bar{x}$ by an arc of M of diameter less than t. There is a point x_j such that $d(\bar{x}, x_j) < \delta_t$, and let β be an arc of M with end-points $\bar{x}$ and x_j and of diameter less than t. The arc β belongs to G and $C(V)$. Since $\bar{x}$ belongs to $K(V)$, the entire arc β belongs to $K(V)$ and hence to H. But this is contrary to the condition that H_{n_j} is a maximal connected subset of $H \cdot G$, since $\bar{x}$ belongs to H_ω.

Then the assumption that H is not a continuous curve has led to a contradiction, which proves our theorem.

THEOREM 7.* *Let K be a bounded continuum which is a subset of a continuous curve M, T be a closed totally disconnected subset of K such that (1) $K - T$ is the sum of two sets K_1 and K_2, (2) if D_i denotes the set of all points $[x]$ such that x belongs to a complementary domain of M that has points of K_i on its boundary then $K_1 + D_1$ and $K_2 + D_2$ are mutually separated sets. If x_1 and x_2 are points of K_1 and K_2 respectively then there exists a simple closed curve J which is a subset of M, separates x_1 and x_2 in S, and such that every point of $J \cdot K$ is a point of T.*

Proof. The set K_1 is closed except for a subset T_1 of T. Let B_i be the point set consisting of the boundaries of the domains of D_i. No point of K_2 is on the boundary of any domain of D_1 and, since only a finite number of the complementary domains of M that belong to D_1 are of diameter greater than any given positive number, every limit point of B_1 which does not belong to B_1 belongs to $K_1 + T_1$. Hence $K_1 + B_1 + T_1$ is closed, and $K_1 + B_1 - T \cdot B_1$ and K_2 are mutually separated. For each point y of $K_1 + T_1 + B_1$, let $r(y) = \frac{1}{2}d(y, K_2)$ and $C(y)$ be the set of all points whose distance from y is less than or equal to $r(y)$. Let H_y be the maximal connected subset of $M \cdot C(y)$ containing y. It is easily seen that $r(y)$ is continuous on $K_1 + B_1$

* In the case where M is the entire plane, this theorem is due to R. G. Lubben, "The Separation of Mutually Separated Subsets of a Continuum by Curves," *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 114 (abstract).

$+ T_1$ and is zero only on $T_1 + T \cdot B_1$, a totally disconnected point set. Then all the conditions of Theorem 6 are satisfied and thus every maximal connected subset of

$$\sum_{y \subset K_1+B_1+T_1} H_y$$

is a continuous curve. Let M_1 be the maximal connected subset of $\sum H_y$ containing x_1. It is evident that no point of K_2 belongs to M_1. Let D' be the complementary domain of M_1 containing x_2 and let B' be its boundary. Suppose B' contains a point z of K_1. There exists a circle $\bar{C}(z)$ with center at z such that every point of M in the interior of $\bar{C}(z)$ lies with z in a connected subset of M which lies within $C(z)$. Since z is approachable from D', there is an arc α one of whose end-points is z and such that $\alpha - z$ is a subset of D'. There is a point w of α such that every point of the subarc zw of α lies within $\bar{C}(z)$. If $< zw$ contains any point of M, this point belongs to M_1 since it belongs to H_z. But this is impossible since $< zw$ belongs to D'. Hence $< zw$ belongs to a complementary domain of M and this domain must be one of the set D_1 since there is a point of K_1 on its boundary. Every point of its boundary belongs to M_1 and hence D' is this domain. But x_2 belongs to no domain of the set D_1. Hence B' contains no point of K_1.

Let J be the boundary of D' with respect to the point x_1.* Then J is the required simple closed curve.

4. In this section we propose to study the properties of a continuous curve M having the property that every two points x and y of the sum of the boundaries of the complementary domains of M may be connected through $x + y$ and $S - M$. This is equivalent to the condition that every two points of the sum of the boundaries of the complementary domains lie together on the boundary of some complementary domain. For the remainder of this paper let W denote the sum of the boundaries of the complementary domains of a continuous curve M.

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* If D is a domain, B is its boundary and x is a point of $S - (D + B)$, the *boundary of D with respect to x* is the boundary of the maximal domain of $S - (D + B)$ containing the point x. If B is a continuous curve, the boundary of D with respect to x is either a simple closed curve or an open curve. Cf. R. L. Moore, "Concerning the Separation of Point Sets by Curves," *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), footnote 5 on page 475. Cf. also my paper, "Concerning the Boundaries of Domains of a Continuous Curve," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 565-571.

THEOREM 8. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then W is a continuous curve.*

Proof. Obviously W is connected. Let P be a limit point of W and let $P_1, P_2, P_3, \cdots$, be a sequence of points of W having P as a sequential limit point and such that $P_1 \neq P$. For every $i > 1$, there is a complementary domain D_i of M having P_1 and P_i on its boundary. Since only a finite number of the complementary domains of a continuous curve whose boundaries contain a given point are of diameter greater than a given positive number, there are only a finite number of distinct domains in the set $[D_i]$. Then there is one domain D_j which has infinitely many of the points P_i on its boundary. Hence P belongs to the boundary of D_j and thus to W. Therefore W is closed.

If W is not a continuous curve, there exist two concentric circles C_1 and C_2 and an infinite sequence of continua, $W_\infty, W_1, W_2, W_3, \cdots$, of W satisfying the conditions of the Moore-Wilder Lemma and the further condition that if P_α and Q_α ($\alpha = \infty, 1, 2, 3, \cdots$) are points of $C_1 \cdot W_\alpha$ and $C_2 \cdot W_\alpha$ respectively then C_1 contains an arc β_1, with end-points X_1 and P_∞ which contains every point P_α and in the order $X_1P_1P_2P_3 \cdots P_\infty$, and C_2 contains an arc β_2 with end-points X_2 and Q_∞ which contains every point Q_α and in the order $X_2Q_1Q_2Q_3 \cdots Q_\infty$. Let the radii of C_1 and C_2 be r_1 and r_2 and let $r_2 - r_1 = r > 0$. Let x, y and z be points of W_∞ whose distances from the center of the circle C_1 are $r_1 + r/6$, $r_2 - r/6$ and $r_1 + \frac{1}{2}r$ respectively. There exists a positive number δ such that any point of M whose distance from x is less than δ can be joined to x by an arc of diameter less than $r/6$, and similar for the point y. There exists an integer k such that W_k contains points x_k and y_k such that $d(x, x_k) < \delta$ and $d(y, y_k) < \delta$, and let α_1 and α_2 be arcs of M with end-points x_k and x and y_k and y respectively and each of diameter less than $r/6$. Let z_{k+1} be a point of W_{k+1} whose distance from the center of C_1 is $r_1 + \frac{1}{2}r$. Then z_{k+1} and z cannot be connected through $z_{k+1} + z$ and $S - M$ since z_{k+1} and z do not lie in the same complementary domain of $W_k + W_{k+2} + \alpha_1 + \alpha_2$. Therefore W is a continuous curve.

THEOREM 9. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then either* (1) *M is the sum of three arcs which have the same end-points and no two of which have any other point in common, or* (2) *every maximal cyclic curve* of W is a simple closed curve.*

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* G. T. Whyburn, "Cyclicly connected continuous curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

Proof. Let C be any maximal cyclic curve of W. Then C contains a simple closed curve J. If $C = J$, our theorem is proved. If $C - J$ is non-vacuous, let T be a maximal connected subset of $C - J$. The set T has at least two limit points on J. If w is a point of T there exist arcs wx and wy which lie in T except for the points x and y which are limit points of T on J. The set $wx + wy$ contains an arc xzy with end-points x and y and such that $< xzy >$ lies in T. We shall show that $M = J + xzy$. Suppose that M contains a point which does not belong to $J + xzy$. Let D denote the complementary domain of $J + xzy$ containing this point and α_1 and α_2 denote the arcs of $J + xzy$ from x to y which form the boundary of D and let α_3 be the other arc of $J + xzy$ from x to y. If every point of D belongs to M, a point u of $<\alpha_1>$ cannot be connected to a point v of $<\alpha_2>$ through $u + v$ and $S - M$. Hence D contains a point which does not belong to M. Since the connected set D contains a point of M and a point of $S - M$, it contains a point p of W. Let q be a point of $<\alpha_3>$. But p and q cannot be connected through $p + q$ and $S - M$. Therefore $M = J + xzy$, and our theorem is proved.

THEOREM 10. *If every two points x and y of W may be connected through $x + y$ and $S - M$, then* (1) *if J is any simple closed curve of W then either the interior or the exterior of J contains no point of W,* (2) *if L is an open curve of W then one of the complementary domains of $S - L$ contains no point of W.*

THEOREM 11.* *If W is bounded and non-vacuous and every two points x and y of W may be connected through $x + y$ and $S - M$, then either* (1) *there is one complementary domain of M whose boundary is W, or* (2) *M consists of three arcs with the same end-points and no two of which have any other point in common.*

Proof. By Theorem 8, the set W is a continuous curve. If W contains no simple closed curve our theorem is proved since $M = W$ and the complementary domain whose boundary is W is $S - M$. Now suppose W contains a simple closed curve J_1. If $W = J_1$, then the exterior of J_1 can contain no point of M and this is the desired complementary domain of M whose boundary is W. If a maximal connected subset of $W - J_1$ has more than one limit

* A result closely related to this has recently been published by G. T. Whyburn. See Theorem 27 of his paper, "Concerning the Structure of a Continuous Curve," *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194.

point on J_1 then, by Theorem 9, condition (2) of our theorem holds. Also, by Theorem 10, either $W - J_1$ lies wholly in the exterior or wholly in the interior of J_1.

Case I. Suppose $W - J_1$ is non-vacuous, the interior of J_1 contains $W - J_1$, and no maximal connected subset of $W - J_1$ has more than one limit point on J_1. Let T be a maximal connected subset of $W - J_1$, let P be its limit point on J_1 and let Q be any other point of J_1. Let $P_1, P_2, P_3, \cdots$ be a sequence of distinct points of T having P as a sequential limit point. For every i there is a complementary domain D_i of M having Q and P_i on its boundary. Since there are only a finite number of complementary domains of M of diameter greater than any given positive number that contain points interior to the simple closed curve J_1, there are only a finite number of the domains $[D_i]$ that are distinct. Then there is one domain D_k that has Q and infinitely many of the points P_i on its boundary. Hence P also belongs to the boundary of D_k. Let J_2 be the boundary of D_k with respect to some point in the exterior of J_1. The simple closed curve J_2 contains P and Q and is a subset of the set consisting of J_1 plus the interior of J_1. If any point of J_2 is in the interior of J_1, then some maximal connected subset of $W - J_1$ has two limit points on J_1. Hence $J_2 = J_1$. Then there is a complementary domain D_k of M which lies in the interior of J_1 and whose boundary contains every point of J_1. Now suppose there is a point x of W which is not a boundary point of D_k. Let J_3 be the boundary of D_k with respect to the point x. The set J_3 is a simple closed curve enclosing x, and $J_3 \neq J_1$. Then J_3 has a point x of W in its interior and a point of J_1 (and thus of W) in its exterior, contrary to Theorem 10. Therefore W is the boundary of D_k.

Case II. Suppose $W - J_1$ is non-vacuous, the exterior of J_1 contains $W - J_1$ and no maximal connected subset of $W - J_1$ has more than one limit point on J_1. In a manner very similar to that used in Case I, we may show that there is a complementary domain of M which lies in the exterior of J_1 and has W as its boundary.

THEOREM 12. *If G is a subset of W which is dense in W and every two points x and y of G may be connected through $x + y$ and $S - M$, then every two points p and q of W can be connected through $p + q$ and $S - M$.*

Proof. Let P and Q be points of W. Let $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ be two sequences of points of G which have P and Q as sequential limit points. For every i there is a complementary domain D_i of M whose

boundary contains P_i and Q_i. There can only be a finite number of the domains D_i which are distinct, and thus there is one domain D_j which has on its boundary infinitely many of the points $[P_i]$ and infinitely many of the set $[Q_i]$. Then both P and Q are boundary points of the domain D_j, and P and Q can be joined through $P+Q$ and $S-M$.

THEOREM 13. *If W is unbounded and every two points x and y of W can be connected through $x+y$ and $S-M$, then either* (1) *there is one complementary domain of M whose boundary is W, or* (2) *M consists of three rays with the same vertex* and no two of which have any other point in common, or* (3) *M consists of an open curve plus an arc which has only its end-points in common with the open curve.*

Proof. Let P be a point of a complementary domain of M and let T be an inversion of the plane with respect to some circle whose center is P. Let $W'=T(W)+P$ and $M'=T(M)+P$. The set W' is the sum of the boundaries of the complementary domains of the continuous curve M'. Since every two points x and y of $T(W)$ may be connected through $x+y$ and $S-M'$, by Theorem 12, every two points of W' have this property. The hypothesis of Theorem 11 is satisfied and hence either (a) W' is the boundary of one of the complementary domains of the continuous curve M', or (b) M' consists of three arcs with the same end-points and no two of which have any other point in common. If situation (a) holds, then there is a complementary domain of M whose boundary is W. If situation (b) holds, there are two possibilities according as P is one of the common end-points or is an interior point † of one of the arcs. Under the first possibility, we get situation (2) of our theorem; under the second, we have situation (3).

THEOREM 14. *If W is non-vacuous and every three points of W lie together on the boundary of some complementary domain of M, then there is a complementary domain of M whose boundary is the set W.*

Proof. The hypothesis of either Theorem 11 or Theorem 13 is satisfied according as W is bounded or unbounded. The curve M cannot consist of three arcs with the same end-points but no two of which have any other point

* An *open curve* is a closed set which is in a continuous (1, 1) correspondence with a straight line. If P is a point of an open curve C, the set consisting of P plus either of the maximal connected subsets of $C-P$ is called a *ray*. The point P is called the *vertex* of the ray.

† A point of an arc which is not an end-point of the arc is called an *interior point* of the arc.

in common, for if x, y and z are interior points of the three arcs and no two belong to the same arc, then there is no complementary domain of M whose boundary contains x, y and z. Similarly situations (2) and (3) of Theorem 13 cannot exist. Therefore W is the boundary of some complementary domain of M.

THEOREM 15. *If every two points x and y of a continuous curve M can be connected through $x + y$ and $S - M$, then $M = W$.*

Proof. Suppose $M - W$ contains a point P. By Theorem 8, the point P is not a limit point of W. Then there exists a circle C with center at P such that every point of C and its interior belongs to $M - W$. Let Q be any point of C. Clearly P and Q cannot be connected through $P + Q$ and $S - M$. But this is contrary to the hypothesis of the theorem.

NATIONAL RESEARCH FELLOW IN MATHEMATICS,
THE UNIVERSITY OF TEXAS.

Admissible Numbers in the Theory of Probability.*

By Arthur H. Copeland.

I. *The fundamental assumptions of the theory of probability.*

We shall show that the fundamental assumptions of the theory of probability are equivalent to the assumption that there exists a number, x, satisfying a denumerably infinite set of independent equations. Thus, although these assumptions may seem exceedingly plausible on the basis of physical considerations, their consistency requires investigation.

The first assumption to which we refer is:

Assumption (a): *If an event has a probability, p, then the ratio of the number of successes, r, to the number of trials, n, approaches the limit, p, as n becomes infinite.*†

For such an event it is proved that the probability of obtaining r successes in n trials is ${}_nC_r p^r (1-p)^{n-r}$. More precisely, it is proved that if $m \cdot n$ trials are made of the given event and these trials are grouped so that the first n trials are in the first group, the second n trials in the second group, etc., and if in s of the m groups there are r successes and $n-r$ failures, then as m becomes infinite the ratio s/m approaches ${}_nC_r p^r (1-p)^{n-r}$. In the derivation of this formula the following assumption is made:

Assumption (b): *the probability of success is a constant and does not vary from one trial to the next.*

The precise meaning of assumption (b) is not obvious. That is, it is not obvious what restrictions this assumption is supposed to place upon the behavior of the given event. However, from the applications which are made of assumption (b), we are justified in concluding that it at least implies that the probability of success of the rth term in any group of n trials is p. In other words, if t be the number of successes of the rth members of the m groups, then t/m approaches p as m becomes infinite.

It is impossible to devise a physical experiment to test the validity of these assumptions, since we are necessarily restricted to a finite number of trials of

* Presented to the American Mathematical Society, Oct. 29, 1927.

† Cf. Guido Castelnuovo, *Calcolo delle probabilita*, prefazione, p. xvi; J. L. Coolidge, *An Introduction to Mathematical Probability*, p. 4.

the event. We can, however, invent a law of behavior for some imaginary event and see whether it is possible for the behavior of such an event to satisfy assumptions (a) and (b). Thus our problem is to construct a model universe of events to test the validity of the fundamental assumptions of the theory of probability.

We shall represent the successes and failures of an event by a sequence of ones and zeroes. These ones and zeroes can be taken as the digits of some number, x. That is, this representation of an event history is also the binary representation of some number, x, lying between zero and one.

Let x represent a given event history, where

$$x = \cdot\ x^1\, x^2\, x^3\, x^4 \cdots x^k \cdots$$

and where x^k is equal to zero or one. Also let

$$p_n(x) = \sum_{k=1}^{n} x^k/n.$$

Then the probability associated with the event history, x, is

$$p(x) = \lim_{n\to\infty} p_n(x)$$

whenever this limit exists.

Next let

$$(r/n)x = \cdot\ x^r x^{r+n} x^{r+2n} \cdots x^{r+kn} \cdots$$

where r, n, and k are integers.* We can now express assumptions (a) and (b) in a very simple form. If an event has a probability, p, then the successes and failures of this event must be represented by a number, x, for which

(1) $\qquad p[(r/n)x]$ is defined

(2) $\qquad p[(r/n)x] = p$

for every pair of integers r, n such that $O < r \leqq n$.

Equations (2) constitute a denumerably infinite set. The equations of this set are not all independent but we shall prove there exists an infinite sub-set of independent equations.†

One other assumption is made in the derivation of the formula ${}_nC_r p^r (1-p)^{n-r}$, namely

(3) $\qquad (1/n)x,\ (2/n)x,\ (3/n)x, \cdots, (n/n)x$ are independent numbers,

* $(r/n)x$ is read r over n of x. It will be observed that $(1/1)x = x$.

† Cf. Section V.

it being understood that a set of numbers is an independent set when and only when the events which these numbers represent are independent.* This assumption is also implied by assumption (b). That is, the probability of success at any trial is p and does not depend upon the success or failure of any other trial.

A number, x, is said to belong to the set $A(p)$, where $0 < p < 1$, if and only if it satisfies conditions (1), (2), and (3). A number is admissible if and only if it belongs to some set, $A(p)$.† If an event has a history which is represented by an admissible number, then the behavior of the event is admissible in the sense that it satisfies the assumptions underlying the theory of probability. The problem of establishing the consistency of these assumptions is identical with that of proving the existence of admissible numbers.

There is one difficulty in the interpretation of our model. We shall see that if we have given an admissible number, we can change arbitrarily a finite number of its digits and obtain in this way a new admissible number. Thus, since any physical experiment involves only a finite number of trials, the assumption that the history of the probable event must be represented by an admissible number, apparently gives no restriction whatsoever on the behavior of physical events. Before attempting to answer this objection let us point out that our model has not introduced a new difficulty but thrown new light on an old one. The above objection has no more justification than the statement that since in a given number of trials of a probable event, all combinations of successes and failures are possible, the assumption that this event has a probability places no restriction on the behavior of the event.

We can imagine that an experimenter makes a series of trials of a given event, then a second experimenter makes another series of trials of the same event, etc. If the object on which the experiments are made, wears out, a similar one is constructed. The work continues indefinitely. We assume that the history thus obtained can be represented by an admissible number. This assumption does restrict the behavior of the event—it restricts what might be described as its asymptotic behavior.

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* Precise definitions of independence are given on pp. 540 and 541.

† It will be observed that an admissible number cannot be rational since the digits of a rational number repeat periodically and such a number is excluded by condition (2).

II. *The Algebra of Event Histories.*

We shall let $x \cdot y$ represent the compound event which consists in the conjunction of the two events, x and y (where x and y are the events whose histories are represented by the numbers x and y). Then

$$x \cdot y = \cdot u^1 u^2 u^3 \cdots u^i \cdots \quad \text{where} \quad u^i = x^i y^i.$$

This product obeys the two laws:

A. $$x \cdot y = y \cdot x$$

B. $$(x \cdot y) \cdot z \quad = x \cdot (y \cdot z).$$

If the events, x and y, are mutually exclusive, then $x \cdot y = 0$. When this is the case, we can represent the history of the event "either x or y" by

$$x + y = \cdot v^1 v^2 v^3 \cdots v^i \cdots \quad \text{where} \quad v^i = x^i + y^i.$$

The expression, $x + y$, is the algebraic sum of the numbers, x and y.

This algebra of event histories also obeys the laws:

C. $$x + y = y + x \qquad \text{if } x \cdot y = 0$$

D. $$(x + y) + z = x + (y + z) \quad \text{if } x \cdot y = y \cdot z = z \cdot x = 0$$

E. $$(x + y) \cdot z \quad = x \cdot z + y \cdot z \quad \text{if } x \cdot y = 0.$$

In the case of two events the theorem of total probability can be stated in the form:

THEOREM 1. *If* $x \cdot y = 0$, *then* $p(x + y) = p(x) + p(y)$.

The extension to the case of n events is immediate.

Let us consider the history

$$1 - x = \cdot w^1 w^2 w^3 \cdots w^i \cdots \quad \text{where} \quad w^i = 1 - x^i.$$

We have the following identity

$$x \cdot (1 - x) = 0.$$

The event $1 - x$ is the complement of the event x. That is, $1 - x$ occurs whenever x fails, and fails whenever x occurs.

The following experiment serves to illustrate the use of this algebra: A penny was flipped ninety times and the heads and tails recorded. An event, x, consisted in the occurrence of a head. The successes and failures of x were arranged in groups of three. An event, X, consisted in the occurrence of two heads in a group of three flips. Hence

$$X = (1/3)x \cdot (2/3)x \cdot [1 - (3/3)x] + (2/3)x \cdot (3/3)x \cdot [1 - (1/3)x] + (3/3)x \cdot (1/3)x \cdot [1 - (2/3)x].$$

It was found that $p_{90}(x) = .466$ and $p_{30}(X) = .400$. These results agree fairly well with the theoretical values, $p(x) = .500$, $p(X) = .375$.

As a consequence of theorem 1, we have the equation

(4) $\quad p(x_1 \cdot x_3 \cdot x_4 \cdots x_n) + p(x_2 \cdot x_3 \cdot x_4 \cdots x_n) = p[(x_1 + x_2) \cdot x_3 \cdot x_4 \cdots x_n]$

if $x_1 \cdot x_2 = 0$.

As special cases of this equation we have

(5) $\quad p(x_1 \cdot x_2 \cdot x_3 \cdots x_n) + p[(1 - x_1) \cdot x_2 \cdot x_3 \cdots x_n] = p(x_2 \cdot x_3 \cdots x_n).$

(6) $\quad p(1 - x) = 1 - p(x).$

The following equations are easily verified:

(7) $$(r/n) \prod_{k=1}^{N} x_k = \prod_{k=1}^{N} (r/n)x_k$$

(8) $$(r/n) \sum_{k=1}^{N} x_k = \sum_{k=1}^{N} (r/n)x_k \text{ if } x_i \cdot x_j = 0 \text{ when } i \neq j$$

(9) $$(r/n)(1 - x) = 1 - (r/n)x$$

(10) $$(r_2/n_2)[(r_1/n_1)x] = (R/N)x$$

where $N = n_1 n_2$ and $R = r_1 + (r_2 - 1)n_1$.

III. *Independent Numbers.*

It is a fundamental theorem that if a compound event consist in the conjunction of n independent events, then the probability of the compound event is the product of the probabilities of the individual events. This theorem is true in the case of two events and can be made true in the general case by formulating an accurate definition of independence.

Let us consider first the case of two events, x and y. The probability that y will occur if x occurs is

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x^i \cdot y^i}{\sum_{i=1}^{n} x^i} = \frac{p(x \cdot y)}{p(x)}$$

whenever $p(x)$ and $p(x \cdot y)$ exist and $p(x) \neq 0$. The probability that y will occur if x fails is $p[(1 - x) \cdot y]/p(1 - x)$. We can now state the following definition: *

* For some purposes it might be convenient to formulate a more restrictive definition of independence, but in the discussion of admissible numbers the above definition is sufficient.

Two events x and y are independent if and only if

$$\frac{p[(1-x)\cdot y]}{p[1-x]} = \frac{p(x\cdot y)}{p(x)} \quad \text{and} \quad \frac{p[(1-y)\cdot x]}{p[1-y]} = \frac{p(x\cdot y)}{p(y)} .$$

It is understood that the probability function is defined for all of the arguments involved and that $p(x)$ and $p(y)$ are different from one or zero.

The theorem of compound probability together with its converse can be stated as follows:

THEOREM 2. *A necessary and sufficient condition that two numbers, x and y, be independent is that*

$$p(x\cdot y) = p(x)\cdot p(y) \quad \text{where} \quad 0 < p(x), p(y) < 1.$$

This theorem is easily proved by means of equations (5) and (6).

It is natural to extend the definition of independence by saying that n events are independent provided each is independent of each of the other $n-1$. Can we, on the basis of this definition, extend theorem 2 to the case of n events? The following example throws considerable light on this question. The histories of three events x_1, x_2, and x_3, are as follows:

$$\begin{aligned} x_1 &= \cdot 1100, 1100, 1100, 1100, \cdots \\ x_2 &= \cdot 1010, 1010, 1010, 1010, \cdots \\ x_3 &= \cdot 1001, 1001, 1001, 1001, \cdots . \end{aligned} \tag{11}$$

We have the equations, $p(x_1) = p(x_2) = p(x_3) = \frac{1}{2}$, and $p(x_1\cdot x_2) = p(x_2\cdot x_3) = p(x_3\cdot x_1) = \frac{1}{4}$, but $p(x_1\cdot x_2\cdot x_3) = \frac{1}{4} \neq p(x_1)p(x_2)p(x_3)$. That is, each of the three events is independent of the other two, but the probability of the compound event consisting in the conjunction of x_1, x_2, and x_3 is not the product of the individual probabilities.

A second example is also instructive. An urn, U_1, contains n balls each of which is half red and half green. It contain also n white balls. An urn, U_2, contains n red balls and n green balls. U_3 contains n blue balls and n yellow balls. U_4 contains n blue and yellow balls and n white balls. Two men, A and B, draw balls one at a time from the urns, replacing the balls after each draw. A draws alternately from U_1 and U_2 and B draws from U_3 whenever A draws from U_1, and from U_4 whenever A draws from U_2. Events x_1, x_2, x_3, x_4 consist in the drawing of balls having respectively the colors red, green, blue, yellow. Each event has the probability, a half, and is independent of each of the others, but the compound event which is the conjunction of the four events, has the probability, zero.

These facts can be more easily seen if we examine the event histories.

Let X_1 be the history of the drawing of a red and green ball from U_1, X_2 that of drawing a red ball from U_2, X_3, that of drawing a blue ball from U_3, and X_4, that of drawing a blue and yellow ball from U_4. Then the equations

$$(12)\qquad \begin{array}{ll} (1/2)x_1 = X_1 & (2/2)x_1 = X_2 \\ (1/2)x_2 = X_1 & (2/2)x_2 = 1 - X_2 \\ (1/2)x_3 = X_3 & (2/2)x_3 = X_4 \\ (1/2)x_4 = 1 - X_3 & (2/2)x_4 = X_4 \end{array}$$

define x_1, x_2, x_3, x_4 in terms of X_1, X_2, X_3, X_4. From these equations we get

$$(13)\qquad \begin{array}{ll} (1/2)(x_1 \cdot x_2) = X_1 & (2/2)(x_1 \cdot x_2) = 0 \\ (1/2)(x_3 \cdot x_4) = 0 & (2/2)(x_3 \cdot x_4) = X_4 \\ (1/2)(x_1 \cdot x_3) = X_1 \cdot X_3 & (2/2)(x_1 \cdot x_3) = X_2 \cdot X_4 \\ (1/2)(x_2 \cdot x_4) = X_1 \cdot (1 - X_3) & (2/2)(x_2 \cdot x_4) = (1 - X_2) \cdot X_4 \\ (1/2)(x_1 \cdot x_4) = X_1 \cdot (1 - X_3) & (2/2)(x_1 \cdot x_4) = X_2 \cdot X_4 \\ (1/2)(x_2 \cdot x_3) = X_1 \cdot X_3 & (2/2)(x_2 \cdot x_3) = (1 - X_2) \cdot X_4. \end{array}$$

Hence we get

$$(14)\qquad x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 0.$$

From these equations it follows that $p(x_i \cdot x_j) = p(x_i)p(x_j)$ if $i \neq j$ and $i, j = 1, 2, 3, 4$, but $p(x_1 \cdot x_2 \cdot x_3 \cdot x_4) = 0$.

From these two examples it is clear that in order to insure the independence of N numbers, $x_1, x_2, \cdots, x_N$, it is not sufficient to assume that every x_i is independent of every x_j (where $i \neq j$). A definition which is sufficient is the following:

*The numbers $x_1, x_2, x_3, \cdots, x_N$, are independent if and only if every number, $x_{\mu_1} \cdot x_{\mu_2} \cdot x_{\mu_3} \cdots x_{\mu_m}$, is independent of every number, $x_{\nu_1} \cdot x_{\nu_2} \cdots x_{\nu_n}$, where the numbers, $x_{\mu_1}, x_{\mu_2}, \cdots, x_{\nu_1}, \cdots, x_{\nu_n}$, are all different and all belong to the set $x_1, x_2, \cdots, x_N$.**

We can now state the general theorem of compound probability.

THEOREM 3. *A necessary and sufficient condition that the distinct numbers, $x_1, x_2, x_3, \cdots, x_N$, be independent is that every subset, $x_{\nu_1}, x_{\nu_2}, x_{\nu_3}, \cdots, x_{\nu_n}$, no two numbers of which are equal, satisfy the condition*

$$(15)\qquad 0 < p(x_{\nu_1} \cdot x_{\nu_2} \cdot x_{\nu_3} \cdots x_{\nu_n}) = p(x_{\nu_1})p(x_{\nu_2})p(x_{\nu_3}) \cdots p(x_{\nu_n}) < 1.$$

Is it not obvious that equations (15) are compatible, and if these equations are incompatible for every N, then there exist no independent events. The following theorems are concerned with the existence of independent events:

* N is understood to be an unrestricted integer.

THEOREM 4. *There exist n independent numbers,* $X_1, X_2, \cdots, X_k, \cdots, X_n$, *such that* $p(X_k) = \frac{1}{2}$.

We shall show the existence of n independent numbers whose digits repeat periodically, the period being 2^n. Let us write the first 2^n digits of these numbers in tabular form as follows.

$$\begin{array}{l} X_1^1\, X_1^2\, X_1^3 \cdots X_1^{2^n} \\ X_2^1\, X_2^2\, X_2^3 \cdots X_2^{2^n} \\ \cdots\cdots\cdots\cdots \\ X_n^1\, X_n^2\, X_n^3 \cdots X_n^{2^n}. \end{array}$$

We shall arrange the columns in $n+1$ groups, $G_0, G_1, G_2, \cdots, G_n$. Group G_k will contain all possible different columns having k zeroes and $n-k$ ones. Hence G_k contains ${}_nC_k$ columns, and the total number of columns in the period is 2^n.

Let us compute the value of $p(X_{v_1} \cdot X_{v_2} \cdot X_{v_3} \cdots X_{v_r})$ where the numbers, $X_{v_1}, X_{v_2}, \cdots, X_{v_r}$, are all different and all belong to the set, $X_1, X_2, \cdots, X_n$. Since the digits of $X_{v_1} \cdot X_{v_2} \cdots X_{v_r}$ repeat periodically, we need only to count the number of ones in a period. The kth group contains ${}_{n-r}C_k$ ones, and hence the total number of ones contributed by all the groups is $\sum_{k=0}^{n-r} {}_{n-r}C_k = 2^{n-r}$. Thus

$$p(X_{v_1} \cdot X_{v_2} \cdot X_{v_3} \cdots X_{v_r}) = 2^{n-r}/2^n = 2^{-r}.$$

In particular $p(X_{v_k}) = \frac{1}{2}$, and therefore, by theorem 3, the numbers $X_1, X_2, \cdots X_n$ are independent.

Before we consider the general problem of the independence of numbers associated with an arbitrary probability, we shall need two more theorems.

THEOREM 5. *If the numbers,* $x_1, x_2, x_3, \cdots, x_N$, *are independent, then the numbers,* $1 - x_1, x_2, x_3, \cdots, x_N$, *are independent.*

We have the equation

$$p[(1-x_1) \cdot x_{v_1} \cdot x_{v_2} \cdots x_{v_n}] = p(x_{v_1} \cdot x_{v_2} \cdots x_{v_n}) - p(x_1 \cdot x_{v_1} \cdot x_{v_2} \cdots x_{v_n}).$$

By hypothesis the numbers, $x_1, x_2, \cdots, x_N$, are independent and hence

$$\begin{aligned} p(x_{v_1} \cdot x_{v_2} \cdots x_{v_n}) &- p(x_1 \cdot x_{v_1} \cdots x_{v_n}) \\ &= p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n}) - p(x_1)p(x_{v_1}) \cdots p(x_{v_n}). \end{aligned}$$

Therefore

$$\begin{aligned} p[(1-x) \cdot x_{v_1} \cdot x_{v_2} \cdot x_{v_n}] &= [1 - p(x_1)]p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n}) \\ &= p(1-x_1)p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n}). \end{aligned}$$

It follows that the numbers, $1 - x_1, x_2, x_3, \cdots, x_N$, are independent.

THEOREM 6. *If the numbers $x_1, x_3, x_4, \cdots, x_N$, are independent, and the numbers $x_2, x_3, x_4, \cdots, x_N$, are independent, and if $x_1 \cdot x_2 = 0$, then the numbers $(x_1 + x_2), x_3, x_4, \cdots, x_N$, are independent.*

By hypothesis

$$p(x_1 \cdot x_{v_1} \cdot x_{v_2} \cdots x_{v_n}) = p(x_1)p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n})$$

and

$$p(x_2 \cdot x_{v_1} \cdot x_{v_2} \cdots x_{v_n}) = p(x_2)p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n}).$$

Adding these equations we get

$$p[(x_1 + x_2) \cdot x_{v_1} \cdot x_{v_2} \cdots x_{v_n}] = p(x_1 + x_2)p(x_{v_1})p(x_{v_2}) \cdots p(x_{v_n}).$$

Hence the numbers $(x_1 + x_2), x_{v_1}, x_{v_2}, \cdots, x_{v_n}$, are independent.

We can now prove the following theorem:

THEOREM 7. *There exist N independent numbers, $X_1, X_2, \cdots, X_i, \cdots, X_N$, such that*

$$p(X_i) = \cdot p^1 p^2 p^3 \cdots p^j \cdots p^s$$

where $p^j = 0$ or 1, and $p^s = 1$.

We shall prove the theorem by induction. Let us assume that there exist N numbers, $Y_1, Y_2, \cdots, Y_N$, such that $p(Y_i) = p$, and K numbers, $X_1, X_2, \cdots, X_K$, such that $p(X_i) = \frac{1}{2}$. Let us assume that the numbers, $Y_1, Y_2, \cdots, Y_N, X_1, X_2, \cdots, X_K$, are all independent and that $K > \nu N$. We shall show that the numbers, $V_1, V_2, \cdots, V_N, X_{\nu N+1}, \cdots, X_K$, are independent, where $V_i = U_i \cdot Y_i + (1 - X_{\nu_i})$ and where $U_i = X_{(i-1)\nu+1} \cdot X_{(i-1)\nu+2} \cdots X_{i\nu}$.

By theorem 3 the numbers, $U_1 \cdot Y_1, U_2 \cdot Y_2, \cdots, U_N \cdot Y_N, X_{\nu N+1} \cdots X_K$, are independent. By theorem 5 the numbers, $1 - X_\nu, U_2 \cdot Y_2, \cdots, U_N \cdot Y_N, X_{\nu N+1} \cdots X_K$, are independent. Hence by theorem 6 the numbers, $V_1, U_2 \cdot Y_2, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, X_{\nu N+1}, \cdots, X_K$, are independent. Similarly $V_1, 1 - X_{2\nu}, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, \cdots, X_K$, are independent and therefore $V_1, V_2, U_3 \cdot Y_3, \cdots, U_N \cdot Y_N, \cdots, X_K$ are independent. Similarly it can be proved that $V_1, V_2, \cdots, V_N, X_{\nu N+1}, \cdots, X_K$ are independent. Furthermore $p(V_i) = \frac{1}{2} + p2^{-\nu}$ and $p(U_i \cdot Y_i) = p2^{-\nu}$. We have seen that it is possible to satisfy our hypotheses for the case $p = \frac{1}{2}$. It follows that we can satisfy these hypotheses for the case $p = \frac{1}{2} + 2^{-\nu-1}$ or $2^{-\nu-1}$ and hence, by induction, for $p = \cdot p^1 p^2 \cdots p^i \cdots p^s$.

THEOREM 8. *There exist N independent events, $X_1, X_2, \cdots, X_N$, such that $p(X_i) = p$, where p is any number such that $0 < p < 1$.*

Let $p = \cdot p^1 p^2 p^3 \cdots p^s \cdots$. For any s we can choose N independent numbers, $X_1^s, X_2^s, \cdots, X_N^s$, such that $p(X_i^s) = \cdot p^1 p^2 p^3 \cdots p^s$. Hence we can find a sequence of numbers

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_s > \cdots$$

such that $\lim_{s\to\infty} \epsilon_s = 0$ and such that

$$| p(X_{v_1}^s \cdot X_{v_2}^s \cdots X_{v_k}^s) - p^k | < \epsilon_s/3$$

$$| p(X_{v_1}^{s+1} \cdot X_{v_2}^{s+1} \cdots X_{1_k}^{s+1}) - p^k | < \epsilon_s/3$$

for every set of numbers, $X_{v_1}^s, X_{v_2}^s, \cdots, X_{v_k}^s$ contained in the set, $X_1^s, \cdots, X_N^s$, and for every set, $X_{v_1}^{s+1}, \cdots, X_{v_k}^{s+1}$, contained in $X_1^{s+1}, \cdots, X_N^{s+1}$. We can choose two sets of integers $m_1, m_2, m_3, \cdots, m_s, \cdots$, and $n_1, n_2, \cdots, n_s, \cdots$, such that

(a) $$| p_n(X_{v_1}^{s+1} \cdot X_{v_2}^{s+1} \cdots X_{v_k}^{s+1}) - p^k | < \epsilon_s/3 \text{ if } n \geqq m_s$$

(b) $$| p_n(X_{v_1}^s \cdot X_{v_2}^s \cdots X_{v_k}^s) - p^k | + (N_s + m_s)/n_s < \epsilon_s/3 \text{ if } n \geqq n_s$$

where $N_s = n_1 + n_2 + n_3 \cdots n_{s-1}$.

Let X_i be such that the digits $N_s + 1$ to N_{s+1} of X_i are the same as the first n_s digits of X_i^s. Then

(c) $$| p_n(X_{v_1} \cdot X_{v_2} \cdots X_{v_k}) - (n_s/n) p_{n_s}(X_{v_1}^s \cdot X_{v_2}^s \cdots X_{v_k}^s) | < (N_s + m_s)/n < \epsilon_s/3$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$.

Combining (b) and (c) we get

(d) $$| p_n(X_{v_1} \cdot X_{v_2} \cdots X_{v_k}) - (n_s/n) p^k | < (\epsilon_s/3)\,[1 + (n_s/n)] < (2/3)\epsilon_s$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$.

But $| (n_s/n) p^k - p^k | = (n - n_s/n) p^k < \epsilon_s/3$. Therefore

(e) $$| p_n(X_{v_1} \cdot X_{v_2} \cdots X_{v_k}) - p^k | < \epsilon_s$$

if $N_{s+1} \leqq n \leqq N_{s+1} + m_s$. We have also the inequality

(f) $$| p_n(X_{v_1} \cdots X_{v_k}) - (n_s/n) p_{n_s}(X_{v_1}^s \cdots X_{v_k}^s) - (n - N_{s+1}/n) p_{n-N_{s+1}}(X_{v_1}^{s+1} \cdots X_{v_k}^{s+1}) | \leqq (N_s/n) < \epsilon_s/3$$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$.

Combining (a), (b) and (f) we get

(g) $$| p_n(X_{v_1} \cdots X_{v_k}) - (n_s + n - N_{s+1}/n) p^k | < (\epsilon_s/3)\,[1 + (n_s + n - N_{s+1}/n)] < \tfrac{2}{3}\epsilon_s.$$

But

(h) $$| p^k - p^k(n_s + n - N_{s+1})/n | = p^k(N_s/n) < \epsilon_s/3$$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$. Therefore

(i) $$| p_n(X_{v_1} \cdot X_{v_2} \cdots X_{v_k}) - p^k | < \epsilon_s$$

if $N_{s+1} + m_s \leqq n \leqq N_{s+2}$.

Therefore $p(X_{v_1} \cdot X_{v_2} \cdot X_{v_3} \cdots X_{v_k}) = p^k$. Hence the numbers, X_1, X_2, $\cdots$, X_N, are independent and $p(X_i) = p$.

IV. *The Existence of Admissible Numbers.*

By means of theorem 3 we can express the condition for admissibility in a more compact form as follows:

THEOREM 9. *A necessary and sufficient condition that a number, x, be admissible is that*

$$0 < p[(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x] = p^k < 1$$

for every positive integer, n and for every set of distinct integers, $r_1, r_2, \cdots, r_i, \cdots, r_k$, such that $0 < r_i \leqq n$.

In order to investigate the existence of admissible numbers we need also the following theorem:

THEOREM 10. *If $p[(r/n)x] = p_r$ for $r = 1, 2, 3, \cdots, n$, then*

$$p(x) = \sum_{r=1}^{n} (p_r/n).$$

We shall now prove the existence of admissible numbers.

THEOREM 11. *Each set, $A(p)$, contains at least one point, x.*

Given any integer, s, we can select $s!$ independent numbers, X_1^s, X_2^s, $\cdots$, $X_r^s, \cdots, X_{s!}^s$, such that $p(X_r^s) = p$. Let U_s be defined by the equations

(a) $$(r/s!)U_s = X_r^s.$$

Let n be any integer less than or equal to s, and let $m = s!/n$. Then m is an integer. It follows from equations (7) and (10) that

(b) $$p\{(q/m)[(r_1/n)U_s \cdot (r_2/n)U_s \cdots (r_k/n)U_s]\} = p(X_{R_1}^s \cdot X_{R_2}^s \cdots X_{R_k}^s) = p^k$$

where $0 < r_i \leqq n$ and $r_i \neq r_j$ if $i \neq j$, and where $R_i = r_i + (q-1)n$. Equation (b) holds for $q = 1, 2, 3, \cdots, m$. Hence by theorem 10,

$$p[(r_1/n)U_s \cdot (r_2/n)U_s \cdots (r_k/n)U_s] = p^k.$$

Let $\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_s > \cdots$ be a sequence such that $\lim_{s\to\infty} \epsilon_s = 0$. We can select a set of integers, $M_1, M_2, \cdots, M_s, \cdots$, such that

(c) $$| p_N[(r_1/n)U_{s+1} \cdot (r_2/n)U_{s+1} \cdots (r_k/n)U_{s+1}] - p^k | < \epsilon_s/3$$

whenever $N \geqq M_s/n$ and $n \leqq s$. We can choose a second set of integers, $N_1, N_2, \cdots, N_s, \cdots$, such that

(d) $$| p_N[(r_1/n)U_s \cdot (r_2/n)U_s \cdots (r_k/n)U_s] - p^k | + (\nu_s + M_s)/N_s < \epsilon_s/3$$

when $N \geqq N_s/n$ and $n \leqq s$, where $\nu_s = N_1 + N_2 \cdots + N_{s-1}$. At the same time the numbers, N_s, can be chosen so that ν_s/n is an integer when $n \leqq s$. Then N_s/n is an integer.

It is understood that conditions (c) and (d) hold for every n such that $n \leqq s$, and for every set of numbers, $r_1, r_2, \cdots, r_k$, such that $r_i \leqq n$ and $r_i \neq r_j$ if $i \neq j$.

Let x be such that the digits $\nu_s + 1$ to ν_{s+1} of x are the same as the digits 1 to N_s of U_s. Then the digits $(\nu_s/n) + 1$ to ν_{s+1}/n of $(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x$ are the same as digits 1 to N_s/n of $(r_1/n)U \cdot (r_2/n)U \cdots (r_k/n)U$. Hence

(e) $$| p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - (N_s/nN)\, p_{(N_s/n)}\{(r_1/n)U_s \cdots (r_k/n)U_s\} | \leqq (Nn - N_s)/nN \leqq (\nu_s + M_s)/N_s < \epsilon_s/3$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n$.

Combining (d) and (e) we get

(f) $$| p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - (N_s/nN)\, p^k | < (\epsilon_s/3)[1 + (N_s/nN)] < \tfrac{2}{3}\epsilon_s.$$

But

(g) $$| p^k - (N_s/nN)\, p^k | = (Nn - N_s)/nN \cdot p^k < (\nu_s + M_s)/N_s < \epsilon_s/3$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n$. Therefore

(h) $$| p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - p^k | < \epsilon_s$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_s + M_s)/n$.

We have also the inequality

(i) $$| p_N\{(r_1/n)x \cdots (r_k/n)x\} - (N/nN)\, p_{(N_s/n)}\{(r_1/n)U_s \cdots (r_k/n)U_s\} - [N - (\nu_{s+1}/n)/N]\; p_{[N-(\nu_{s+1}/n)]}\{(r_1/n)U_{s+1} \cdots\} | \leqq \nu_s/nN \leqq \epsilon_s/3$$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$.

Combining (c), (d) and (i) we get

(j) $\quad | p_N\{(r_1/n)x \cdots (r_k/n)x\} - [N/nN + (nN - \nu_{s+1})/nN]\, p^k | < \frac{2}{3}\epsilon_s.$

But

(k) $\quad | p^k - [N_s/nN + (nN - \nu_{s+1})/nN]\, p^k | < \epsilon_s/3$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$. Therefore

(l) $\quad | p_N\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)x\} - p^k | < \epsilon_s$

if $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$.

It follows from (h) and (l) that

$$p\{(r_1/n)x \cdot (r_2/n)x \cdots (r_k/n)\} = p^k.$$

Hence x is a point of the set $A(p)$.

THEOREM 12. *Each set, $A(p)$, has the power of the continuum.*

Given any s, we can find $2(s!)$ independent numbers, $X_1^s, X_2, \cdots, X_{s!}^s$, $Y_1^s, \cdots, Y_{s!}^s$, such that $p(X_r^s) = p(Y_r^s) = p$. Let U_s and V_s be defined by the equations

$$(r/s!)U_s = X_r^s \qquad (r/s_s)V_s = Y_r^s$$

where $r = 1, 2, 3, \cdots, s!$. We can define two sets of integers, M_s and N_s, such that conditions (c) and (d) of theorem 11, hold for both sets of numbers, U_s and V_s.

Let a be any number such that $0 \leqq a \leqq 1$. Then $a = \cdot a^1 a^2 a^3 \cdots a^s \cdots$ where $a^s = 0$ or 1. Let us define a number, x, as follows. The digits $\nu_s + 1$ to ν_{s+1} of x are the same as the digits 1 to N_s of U_s or V_s according as a^s is one or zero. Then x belongs to $A(p)$. Moreover, there is a one to one correspondence between the numbers, x, and the numbers, a. Hence $A(p)$ has the power of the continuum.

Another property of the sets, $A(p)$, is the following: If x belongs to the set, $A(p)$ and if $y = \sum_{k=1}^{n} a^k 2^{-k} + x \cdot 2^{-n}$ where $a^k = 0$ or 1, then y belongs to the set, $A(p)$. Conversely, if y belongs to the set, $A(p)$, then x does also. That is, if the interval $(0, 1)$, be divided into 2^n equal parts, then the points of $A(p)$ in each of the sub-intervals, form an image of the whole set, $A(p)$. The numbers, x and y, are related by the equation $(n + 1/1)y = x$. Thus we have to prove the following theorem:

THEOREM 13. *$(N/1)x$ belongs to $A(p)$ if any only if x belongs to $A(p)$.*

Let us assume that s belongs to $A(p)$ and prove that $(N/1)x$ belongs to $A(p)$. We have the equation

$$p\{r_1/n)[(N/1)x]\cdot(r_2/n)[(N/1)x]\cdots(r_k/n)[(N/1)x]\} = p[(R_1/n)x\cdot(R_2/n)x\cdots(R_k/n)x]$$

where $R_i = N + r_i - 1$, $r_i \leqq n$ and $r_i \neq r_j$ if $i \neq j$.

Case I. If $(N-1)/n$ is an integer, then we can let $(N-1)=(\rho-1)n$ where ρ is an integer. Hence

$$p[(R_1/n)x\cdot(R_2/n)x\cdots(R_k/n)x] = p[(\rho/1)\{(r_1/n)x\cdots(r_k/n)x\}]$$
$$p[(r_1/n)x\cdot(r_2/n)x\cdots(r_k/n)x] = p^k.$$

Case II. If $N-1 \leqq n$, then $R_i \leqq 2n$ and $R_i \neq R_j$ if $i \neq j$. Let q and m be two integers such that $0 < q < m$. Then

$$p\{q/m)[(R_1/n)x\cdots(R_k/n)x]\} = p[(Q_1/M)x\cdots(Q_k/M)x] = p^k$$

where $M = n\cdot m$ and $Q_i = R_i + (q-1)n \leqq M$. Given any positive number, ϵ, we can first choose m so that $1/m < \epsilon/2$ and then choose a number, μ_0, so that

$$p_k - (\epsilon/2) < p_\mu\{(Q_1/M)x\cdots(Q_k/M)x\} < p^k + (\epsilon/2)$$

whenever $\mu \geqq \mu_0$, where $q = 1, 2, 3, \cdots, m-1$. Then

$$p^k - \epsilon < [(m-1)/m]p^k - (\epsilon/2) < p_\nu\{(R_1/n)x\cdots(R_k/n)x\} < p^k + (1/m) + (\epsilon/2) < p^k + \epsilon$$

whenever $\nu \geqq m\mu_0$. That is

$$p\{(R_1/n)x\cdots(R_k/n)x\} = p^k.$$

Case III. If $N-1 > n$ and $(N-1)/n$ is not an integer, we can find two numbers, N' and N'' such that $(N'-1)/n$ is an integer $N''-1 \leqq n$ and $N = N' + N'' - 1$. Then $(N/1)x = (N''/1)\{(N'/1)x\}$ and

$$p^k = \{(r_1/n)x\cdots(r_k/n)x\}$$
$$p[(r_1/n)\{(N'/1)x\}\cdots(r_k/n)\{(N'/1)x\}]$$
$$p[(r_1/n)\{(N/1)x\}\cdots(r_k/n)\{(N/1)x\}].$$

Therefore $(N/1)x$ belongs to the set, $A(p)$.

Next let us start with the assumption that $(N/1)x$ belongs to $A(p)$. Then $(N'/1)\{(N/1)x\} = [(N+N'-1)/1]x$ belongs to $A(p)$. Let us choose N' so that $N + N' - 2 = (\rho-1)n$ where ρ is an integer. Then

$$p^k = p[(r_1/n)\{[(N+N'-1)/1)x]\}\cdots(r_k/n)\{[(N+N'-1)/1]x\}]$$
$$= p[(\rho/1)\{(r_1/n)x\cdot(r_2/n)x\cdots(r_k/n)x\}]$$
$$= p[(r_1/n)x\cdots(r_k/n)x].$$

Therefore x belongs to $A(p)$.

COROLLARY. *If x belongs to $A(p)$ we can change arbitrarily a finite number of the digits of x and obtain in this way a new number, y, which belongs to $A(p)$.*

THEOREM 14. *Every point of the interval $(0, 1)$ is a point of condensation of each of the sets, $A(p)$.*

This theorem is an immediate consequence of theorems 12 and 13.

From theorem 14 it follows that between any two distinct points of the interval $(0, 1)$, each of the non-denumerable infinity of non-overlapping sets, $A(p)$, has a non-denumerable infinity of points.

V. *Independence of the conditions for admissibility.*

Equations (2) are not all independent. In fact, the equations $p\{(r/n)x\} = p$, where $r = 1, 2, 3, \cdots, n$, imply the equation, $p(x) = p$. Moreover, if n is the product of two integers, n_1, n_2, then since

$$(r_2/n_2)\{(r_1/n_1)x\} = [(r_1 + (r_2 - 1)n_1)/n_1n_2]x$$

and

$$(r_1/n_1)\{(r_2/n_2)x\} = [(r_2 + (r_1 - 1)n_2)/n_1n_2]x,$$

the equations $p\{(r/n)x\} = p$ imply the equations, $p\{(r_1/n_1)x\} = p$ and $p\{(r_2/n_2)x\} = p$, where $r_1 = 1, 2, \cdots, n_1$ and $r_2 = 1, 2, \cdots, n_2$.

However, the equations of the set, $p\{(r/n)x\} = p$ (where n is any prime number and $0 < r < n$), are independent. These equations constitute an infinite sub-set of equations (2). To establish their independence, it will be sufficient to prove the following theorem:

THEOREM 15. *Given an arbitrary prime number, N, and an integer, Q, such that $0 < Q < N$, there exists a number, x, such that*

$$p\{(Q/N)x\} \neq p$$

but $\quad p\{(R/N)x\} = p$ *if* $0 < R < N$ and $R \neq Q$

and $\quad p\{(r/n)x\} = p$ *if* $0 < r < n$

where n is an arbitrary prime number distinct from N.

Let X_R belong to the set, $A(p)$, if $0 < R < N$ and $R \neq Q$. Let X_Q and X_N belong respectively to $A(p_1)$ and $A(p_2)$, where $(p_1 + p_2)/2 = p$ and $p_1, p_2 \neq p$. Let x be defined by the equations

$$(R/N)x = X_R \text{ where } R = 1, 2, \cdots, N.$$

Then

$$p\{(Q/N)x\} = p_1 \neq p \quad \text{and} \quad p\{(R/N)x\} = p$$

if $0 < R < N$ and $R \neq Q$.

We have to show that $p\{(r/n)x\} = p$ if $0 < r < n$ and n is any prime number such that $n \neq N$. Given the numbers, n, N, r, R, there exist two numbers, r' and R', such that

$$(R/N)\{(r/n)x\} = (r'/n)\{(R'/N)x\}$$

The pair of numbers, r' and R', is a solution of the equation,

(a) $$r + (R-1)n = R' + (r'-1)N \qquad \text{where}$$

(b) $$0 < r' < n \quad \text{and} \quad 0 < R' < N.$$

The solution of (a) and (b) is unique, and since n and N are prime and $n \neq N$, corresponding to two distinct values of R there are two distinct values of R'. Hence as R takes on the values, 1, 2, 3, $\cdots$, N, R' also takes on the values, 1, 2, $\cdots$, N (but in different order). Thus, since

$$p[(R/N)\{(r/n)x\}] = p[(r'/n)X_{R'}] = p(X_{R'}) \quad .$$

it follows from theorem 10 that

$$p\{(r/n)x\} = [((N-2)p + p_1 + p_2)/N] = p.$$

We have now proved that every admissible number must satisfy an infinite set of independent equations.

VI. *Sufficiency of the assumptions.*

In selecting our assumptions we have been using as a guide the formula, ${}_nC_r p^r (1-p)^{n-r}$. It is easily seen that our assumptions are such that this formula is verified. That is, if an event, x, is such that the number, x, belongs to $A(p)$, and if the trials of x are grouped in groups of n (in the manner previously described), then the event, X, which consists in the occurrence of r successes in a group of n, is such that $p(X) = P = {}_nC_r p^r (1-p)^{n-r}$.

However, we usually mean more than this when we say that an event, X, has the probability, P; we mean that X belongs to the set, $A(P)$. The fact that X does belong to the set, $A(P)$, is a consequence of the following theorem:

THEOREM 16. *If x belongs to the set, $A(p)$ and if $X = \sum_{j=1}^{M} Y_j$ and*

$$Y_j = \prod_{i=1}^{r_j} (r_{ij}/n)x \cdot \prod_{i=r_j+1}^{n} \{1 - (r_{ij}/n)x\}.$$

where $0 < r_{ij} \leqq n$ *and* $r_{i'j} \neq r_{ij}$ *if* $i' \neq i$ *and where* $Y_{j'} \neq Y_j$ *if* $j' \neq j$; *then* X *belongs to* $A(P)$, *where* $P = \sum_{j=1}^{M} p^{r_j}(1-p)^{n-r_j}$.

By hypothesis $Y_{j'} \cdot Y_j = 0$ if $j' \neq j$, so that X is defined. We have to prove that

$$p\{\prod_{k=1}^{L} (\rho_k/\nu)X \cdot\} = P^L \text{ where } 0 < \rho_k \leqq \nu.$$

Let $Y_j{}^k = (\rho_k/\nu)Y_j)$ then

$$Y_j{}^k = \prod_{i=1}^{r_j} (R_{ij}{}^k/N)x \cdot \prod_{i=r_j+1}^{n} \{1-(R_{ij}{}^k/N)x\}.$$

where $N = n\nu$ and $R_{ij} = r_{ij} + (\rho_k - 1)n$. Hence $Y_j{}^k \cdot Y_{j'}{}^k = 0$ if $j' \neq j$. Therefore

$$\prod_{k=1}^{L} [(\rho_k/\nu)X] = \prod_{k=1}^{L} [\sum_{j=1}^{M} Y_j{}^k] = \Sigma [\prod_{k=1}^{L} Y_{jk}{}^k]$$

where the expression on the right is the formal algebraic expansion of the expression in the middle.

Since the numbers of each set, $Y_{j_1}{}^1, Y_{j_2}{}^2, \cdots, Y_{j_M}{}^M$, are independent, we have

$$p[\prod_{k=1}^{L} \{(\rho_k/\nu)X\}] = \Sigma \{\prod_{k=1}^{L} p(Y_{jk}{}^k)\} = \prod_{k=1}^{L} [\sum_{j=1}^{M} p(Y_j{}^k)]$$

But

$$\sum_{j=1}^{M} p(Y_j{}^k) = p\{(\rho_k/\nu)X\} = p(X) = P.$$

Therefore

$$p[\prod_{k=1}^{L} \{(\rho_k/\nu)X\} = P^L,$$

and hence X belongs to $A(P)$.*

If the summation, $\sum_{j=1}^{M} Y_j$, includes all the possible numbers, Y, for which $z_1 \leqq (r_j/n) \leqq z_2$, then the event, X, consists in the occurrence of a success ratio lying between z_1 and z_2, in n trials of the event, x. If the summation, $\sum_{j=1}^{L} Y_j$ includes all numbers, Y_j, for which $r_j = r$, then $M = {}_nC_r$ and X consists in the occurrence of r successes in n trials of x. In either case the assumptions about the event, x, are such that the related event, X, is a probable event.

Even though the assumptions (a) and (b) demand more, or at least as

* It follows from Theorem 16 that every combination of ones and zeroes must be displayed infinitely many times as successive digits of every admissible number.

much as (1), (2), (3), the former seem natural and the latter, somewhat artificial. The method of grouping defined in assumptions (1), (2), (3) is a simple one which would very likely be chosen if, for example, one wished to test experimentally the formula, ${}_nC_r p^r (1-p)^{n-r}$. On the other hand there is no physical reason why other methods of grouping should not work equally well.

Suppose we choose a set of positive integers, $\lambda_1 < \lambda_2 < \lambda_3 \cdots \lambda_n \cdots$ and make λ_n trials of an event, x, with probability, p. If we record the trials numbered λ_1, λ_2, etc., and if n is large, we should expect approximately pn successes of the recorded trials. We should also expect the success ratio to approach p as n becomes infinite. That is, if y is such that $y^i = x^{\lambda_i}$ then we should expect to have the equation, $p(y) = p$. But this is impossible for every set of numbers, $\lambda_1, \lambda_2, \lambda_3, \cdots$, since in particular a set might be chosen (even in advance) so that $y = 0$. It seems obvious that such a choice is not probable. On the other hand it might be probable that y should not be admissible. Let us consider the following theorem.

THEOREM 17. *If $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \cdots$ is any set of positive integers, and if y is defined by the equations*

$$(R/N)y = (\lambda_R/\lambda_N)x$$

where x belongs to the set, $A(p)$; then y belongs to $A(p)$.

We have to prove that $p\{(r_1/n)y \cdot (r_2/n)y \cdots (r_k/n)y\} = p^k$. We have the equation, $(R/N)\{r_i/n)y\} = (r_i'/n)\{(R_i'/N)y\}$ where r_i' and R_i' are two integers satisfying the equation, $r_i + (R-1)n = R_i' + (r_i'-1)N$ and such that $0 < r' < n$ and $0 < R' < N$. Hence

$$(R/N)\{r_i/n)y\} = (\rho_i/\nu)x \quad \text{where} \quad \nu = n\lambda_N \quad \text{and}$$

$\rho_i = \lambda_{R_i}' + (r_i'-1)\lambda_N$ and where $\rho_i \neq \rho_j$ if $i \neq j$. Therefore

$$p\{(r_1/n)y \cdot (r_2/n)y \cdots (r_k/n)y\} = p\{(\rho_1/\nu)x \cdot (\rho_2/\nu)x \cdots (\rho_k/\nu)x\} = p^k.$$

It follows from Bernoulli's theorem and from theorem 17 that if N is large, then it is very probable that $\sum_{i=1}^{N} (x^{\lambda_i}/N)$ will differ from $p(x)$ by a small amount.* Thus if an event has an admissible history, its behavior is in accordance with what should be expected of a probable event.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

* Jacobus Bernoulli, *Ars Conjectandi*, Basle (1713), p. 236.

The Differential Invariants of Inversive Geometry.

By Boyd C. Patterson.

1. *The inversive curvature.* The idea of an intrinsic equation in geometries other than ordinary geometry seems to have originated with Pick about 1906.* In 1914 Nohel † published a paper in which he calculated the differential and integral invariants for all geometries under the plane transformation groups, but he did not consider explicitly what is now known as inversive geometry—that is, the geometry defined by the six parameter group of transformations

$$Z = (az + b)/(cz + d)$$

together with

$$w = \bar{Z}$$

The bilinear transformation constitutes, strictly speaking, the homographic group and only when we add an inversion in a circle, or in particular a reflexion in the axis of reals, do we have the complete inversive group as given above. The homography contains an even number of inversions and is a subgroup of the inversive group.

G. W. Mullins in a Columbia dissertation of 1917 gives the differential invariant of lowest order under the inversive group expressed (1) in rectangular coördinates and (2) in terms of the ordinary invariants r, the radius of curvature, and s, the arc-length.

Liebmann,‡ Kubota,§ and Morley ¶ later calculated, by different methods, this invariant which we shall call the inversive curvature in analogy with the corresponding differential invariant of ordinary geometry. We shall now

* Georg Pick, "Natürliche Geometrie ebener Transformationsgruppen," *Sitzungsberichte der Akademie der Wissenschaften*, Wien, Bd. 115 (1906), S. 139.

† Emil Nohel, "Zur natürlichen Geometrie ebener Transformationsgruppen," *Ibid.*, Bd. 123 (1914), S. 2085.

‡ Liebmann, "Beitrage zur Inversionsgeometrie der Kurven," *Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München*, Heft I (1923), S. 79.

§ Kubota, "Beitrage zur Inversionsgeometrie," *Tohoku Imperial University, Science Reports* 13, (1924-25), p. 243.

¶ Morley, "On Differential Inversive Geometry," *American Journal of Mathematics*, Vol. 48 (1926), p. 144.

develop a simple expression for the same—an expression by means of which the inversive curvature may easily be calculated for any curve given parametrically. Later sections will be devoted to applications of this formula.

Under the group of homographies,

$$Z = (az + b)/(cz + d),$$

a subgroup of the inversive group, we have as a seminvariant of a curve expressed as a complex function of a real parameter μ, $z = z(\mu)$, the Schwarzian derivative of z with respect to μ:

$$\{z, \mu\} = \frac{d}{d\mu}\left(\frac{d^2z/d\mu^2}{dz/d\mu}\right) - \frac{1}{2}\left(\frac{d^2z/d\mu^2}{dz/d\mu}\right)^2.$$

By a seminvariant we mean an expression invariant under bilinear transformations of the dependent variable but not invariant under changes of the parameter. The real and imaginary parts of $\{z, \mu\}$ give us in fact two real seminvariants.

We shall use one of these seminvariants to fix a parameter λ (which we call the inversive arc-length) and the other seminvariant when expressed in terms of λ will become a true differential invariant.* We shall, in fact, so fix the parameter λ that $\{z, \lambda\}$ will have its imaginary coefficient equal to 1. The real part of $\{z, \lambda\}$ which is now a true differential invariant we call I_5 for reasons which will appear later.

We have here fixed λ by making the imaginary coefficient of $\{z, \lambda\}$ equal to 1. Might we not have fixed λ so that the real part of $\{z, \lambda\}$ is 1 and the imaginary coefficient the differential invariant? It will be seen in the following paragraph that this cannot be done since λ would then be the solution of a third order differential equation and the number of arbitrary constants would be too great; whereas λ so chosen that

$$\{z, \lambda\} = I_5 + i$$

involves only one arbitrary constant which can easily be disposed of by a suitable selection of the origin of arc-lengths.

Under change of parameter from μ to λ, λ being a real function of μ, we have the formula, due to Cayley, for change of the Schwarzian:

$$\{z, \mu\} = \lambda'^2\{z, \lambda\} + \{\lambda, \mu\},$$

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* Thus in ordinary geometry if a curve is expressed in terms of the ordinary arc-length, s, then the seminvariant $(dx/ds)^2 + (dy/ds)^2$ has the value 1.

primes denoting differentiation with respect to μ. Putting now

$$\{z, \mu\} = A + iB$$

and

$$\{z, \lambda\} = I_5 \pm i$$

we have

$$A + iB = \lambda'^2(I_5 \pm i) + \{\lambda, \mu\},$$

whence λ, the integral invariant, is defined by

$$\lambda = \int (\pm B)^{\frac{1}{2}}\, d\mu, \tag{1.1}$$

the sign being chosen so that λ is real, and I_5, the differential invariant, by

$$I_5 = (A - \{\lambda, \mu\})/B.$$

Since now

$$\pm B = \lambda'^2$$

we can easily calculate $\{\lambda, \mu\}$ in terms of B and its derivatives with respect to μ. It is found to be

$$\{\lambda, \mu\} = (4BB'' - 5B'^2)/8B^2$$

whence

$$I_5 = (8AB^2 - 4BB'' + 5B'^2)/8B^3. \tag{1.2}$$

Given now any curve expressed as a complex function of a real parameter, μ, we find the real and imaginary parts of the Schwarzian $\{z, \mu\}$, and the inversive curvature can be calculated by means of (1.2).

To reduce formula (1.2) to an expression involving the ordinary differential and integral invariants, r and s, it is necessary to make use of the relation

$$(d^2z/ds^2)/(dz/ds) = i/r.$$

This may be easily verified by writing $z = x + iy$ and recalling that $(dx/ds)^2 + (dy/ds)^2 = 1$.

If now we assume a curve to be given as a complex function of the real parameter s, the real and imaginary parts of $\{z, s\}$ are found to be $1/2r^2$ and $-(dr/ds)/r^2$ respectively. Making these substitutions for A and B in (1.2) we have I_5 in terms of the ordinary invariant r and its derivatives with respect to s—viz.

$$I_5 = [4r^2(dr/ds)(d^3r/ds^3) - 5r^2(d^2r/ds^2)^2 - 4r(dr/ds)^2(d^2r/ds^2) + 4(dr/ds)^4 - 4(dr/ds)^2]/8(dr/ds)^3. \tag{1.3}$$

Since r is of the second order in rectangular coördinates, I_5 is of the fifth order. Hence the notation I_5.

Another convenient form of the inversive curvature is to have it expressed in terms of the homologic (equiform) invariants ρ $(= dr/ds)$ and ϕ $(= \int ds/r)$. Making use of (1.3) and the relations

$$dr/ds = \rho\,;\ d^2r/ds^2 = d\rho/rd\phi\,;\ d^3r/ds^3 = (d^2\rho/d\phi^2 - \rho d\rho/d\phi)/r^2$$

and indicating differentiation with respect to the homologic integral invariant, ϕ, by primes we obtain

$$I_5 = (4\rho\rho'' - 5\rho'^2 - 8\rho'\rho^2 + 4\rho^4 - 4\rho^2)/8\rho^3. \tag{1.4}$$

We have previously defined λ as a function of any real parameter, μ, in terms of which a curve may be given. If now the parameter of the curve is the ordinary arc-length, s, since B, the imaginary coefficient of the Schwarzian, is in this case $-(dr/ds)/r^2$ we have

$$d\lambda^2 = \pm(dr/ds)\,ds^2/r^2 = \pm drds/r^2 = \pm dkds$$

where $k = 1/r$ is the ordinary curvature. Similarly the inversive arc-length may be defined in terms of ρ and ϕ:

$$d\lambda^2 = \pm \rho d\phi^2.$$

We have now the fundamental invariants of differential inversive geometry—the differential invariant I_5 and the integral invariant λ. To obtain differential invariants of higher order we differentiate I_5 with respect to λ. That is,

$$I_n = dI_{n-1}/d\lambda.$$

In the foregoing we have been concerned exclusively with invariants under the group of homographies, speaking of them however as inversive invariants. And this is perfectly proper, for we notice that since expressions invariant under all inversions must be invariant under homographies and also the transformation

$$w = \bar{z}$$

(which is simply a reflexion in the real axis) and since I_5 and λ are both real they are therefore invariant under the latter transformation as well as under the bilinear transformation.

At points where the differential of arc becomes zero, B, the imaginary coefficient of $\{z, \mu\}$, may change sign. Thus the inversive arc-length asso-

ciated with a portion of a given curve between two points M and N may be a complex number if there is between M and N a circular point, i. e. a point where $k = 1/r$ is constant or $dk = 0$. For if we call such a point P, the number associated with the arc-length MP is real, but on passing through P the number associated with the arc PN becomes pure imaginary. The sum then of the inversive arc-lengths MP and PN is complex.

To avoid this difficulty we shall adopt the convention previously mentioned of defining λ as in (1. 1) and choosing the sign under the radical so that λ is *always* real.

A relation between the differential invariant I_5 and the integral invariant λ of a curve will be spoken of as an inversive intrinsic equation of the curve. All curves having the same inversive intrinsic equation are inversively equivalent—that is, they are the same or may be made the same by an inversive transformation.

The next question which arises is: What are the curves of constant inversive curvature? These curves are known to be the so-called "double logarithmic spirals. If we were concerned here with the theory of groups we might easily find them as path curves of one-parameter subgroups of the homographic group. As it is, however, we will content ourselves with an examination of the differential equation obtained by equating (1. 4) to a constant, which shows $\rho = \text{const.}$ to be one solution. Hence the homologic intrinsic equation of the curves corresponding to this solution is

$$\rho = a$$

and since $\rho = dr/ds$ the ordinary intrinsic equation is that of a logarithmic spiral, viz.

$$r = as + b.$$

Making $b = 0$ by shifting the origin of arc-lengths we have

(1. 5) $$r = as.$$

We find now from (1. 3)

$$I_5 = (a^2 - 1)/2a,$$

the inversive intrinsic equation of the logarithmic spiral. In this discussion we prefer to speak of loxodromes rather than logarithmic spirals. By definition a loxodrome is a curve cutting arcs of circles from a point to another under a constant angle.

By an inversive transformation such that zero and infinity are sent into two finite points of the plane, m and n, the loxodrome whose ordinary intrinsic equation is (1. 5) becomes a "double logarithmic spiral" or as we prefer to call it, simply a loxodrome. Inversively there is no difference between these two loxodromes; each has two poles, in the one they are at zero and infinity and in the other at two finite points, m and n.

The loxodrome given by the ordinary intrinsic equation (1. 5) has as its equation in polar coördinates

$$R = ce^{a\theta}$$

where $a = \cot \alpha$, α being the constant angle of the loxodrome. This gives us a simple expression for I_5, the inversive curvature:

$$I_5 = \cot 2\alpha.$$

We see that for a loxodrome of angle α all points have the same inversive properties for I_5 has the same value at all points of the curve.

The geometrical interpretation of the differential invariant I_5 is that

$$I_5 - \cot 2\alpha = 0 \tag{1. 6}$$

is the differential equation of the ∞^5 family of loxodromes with fixed angle α.* Given a curve Γ, it may happen that there are points, P, on it at which Γ is very much like the loxodrome; which means that there the loxodrome has fifth order contact. The condition for this is that at P the value of the differential invariant I_5 of Γ must be equal to $\cot 2\alpha$ where α is the angle of the loxodrome. Furthermore, since for all loxodromes $I_6 = 0$, the condition that there be points of Γ at which some loxodrome hyperosculates is that at certain points of Γ, $I_6 = 0$. The angle α of the hyperosculating loxodrome in this case is determined by the equation

$$\alpha = \tfrac{1}{2} \operatorname{arccot} I_5]_P.$$

2. *The differential invariants of the biquadratic.* We shall now investigate the differential inversive invariants of the biquadratic curve, taking up first the case of a biquadratic with a double-point, then a biquadratic with a cusp, and finally the general biquadratic.

If a biquadratic should have a double-point that double-point may be sent

* G. W. Mullins, *Dissertation on Differential Inversive Invariants*, Columbia, 1917.

by an inversive transformation to infinity and we will then have what is customarily called an hyperbola; a rectangular hyperbola is the inverse of a biquadratic with a double-point in which branches cut each other at right angles.

Expressed as a complex function of a real parameter the hyperbola is given by

$$z = \mu + t/\mu$$

where μ is the real parameter, and t, a complex number of absolute value unity, is constant. For $t = i$ we have the parametric representation of a rectangular hyperbola and we shall take this as our example of a nodal biquadratic. The values of μ corresponding to the double-point are $\mu = 0, \infty$.

Upon calculating the Schwarzian $\{z, \mu\}$ and substituting its real and imaginary parts in formula (1.2) we find

$$I_5 = -[5\mu^2/3(\mu^4-1)^3][3(\mu^4+1)^2 - 8\mu^4].$$

Differentiating I_5 with respect to the integral invariant

$$\lambda = 6^{\frac{1}{2}} \int \frac{(\mu^4-1)^{\frac{1}{2}}}{\mu^4+1}\, d\mu \text{ we obtain}$$

$$I_6 = \frac{10\mu}{6^{\frac{1}{2}}(\mu^4-1)^{9/2}}\ (\mu^4+1)^4,$$

and in the same manner

$$I_7 = \frac{5(\mu^4+1)^4}{3(\mu^4-1)^6}\ [(\mu^4+1)^2 + 32\mu^4].$$

An examination of these invariants shows that

$I_5 = 0$ when $\mu = 0, \infty$.
$I_6 = 0$ when $\mu = 0, \infty$.
I_7 never vanishes for real μ.

Now for $\mu = 0, \infty$; $z = \infty$. And at this point alone (the double-point) $I_6 = 0$. From the results of Section 1 we know that this is the condition that some loxodrome hyperosculates the curve at $z = \infty$; and it is precisely that loxodrome for which $I_5 = 0$, i. e. a 45° loxodrome. At the double-point then of a biquadratic which cuts itself orthogonally we find the curve behaving very much like a 45° loxodrome.

A cuspidal biquadratic may be inverted with the center of inversion at the cusp. This inversion will put the cusp at infinity and the curve thus inverted becomes a parabola. The parametric equation of the curve is and

$$z = 2\mu + i\mu^2,$$

and we find its invariants to be as follows:

$$I_5 = (5/24\mu^3)\ [3(\mu^2-1)^2-4]$$
$$I_6 = (5/8\cdot 3^{1/2}\cdot \mu^{9/2})\ (1+\mu^2)^3$$
$$I_7 = (-5/16\mu^6)\ (\mu^2+1)^3\ (\mu^2-3)$$

and $I_5 = 0$ when $\mu = \pm(1+2/3^{1/2})^{1/2}$
I_6 never vanishes for real μ
$I_7 = 0$ when $\mu = \pm 3^{1/2}$.

The map equation of the general biquadratic is given by the Weierstrassian p-function,

$$z = p(u), \tag{2.1}$$

with a rectangular network. For such a network the coefficients of the p-function are real and hence

$$\bar{z} = \bar{p}(\bar{u}) = p(\bar{u}).$$

It is shown in Harkness and Morley *Theory of Functions,* p. 336 that as u varies along a fixed horizontal line the map in the z-plane is a Cartesian curve consisting of two ovals. When it is necessary for us to bring into prominence the fact that the variable u moves along a horizontal line we shall write u in the form $\alpha + i\beta_0$, thus indicating that the imaginary part of the variable u is fixed. Limiting u in this manner, the complete Cartesian mapped by (2.1) includes the representation of four horizontal lines of the rectangle of periods; the lines $i\beta_0$, $2\omega_2 - i\beta_0$ correspond to the outer oval and the lines $\omega_2 \pm i\beta_0$ to the inner oval, $2\omega_1$ and $2\omega_2$ being the real and imaginary periods of the network.

A Cartesian is a biquadratic one of whose foci is at infinity. Since foci invert into foci any general biquadratic may be sent into a Cartesian. The three real foci of the curve (2.1) are e_1, e_2, e_3, where $e_i = p(\omega_i)$.

When u varies along a quarter line of the rectangle, i. e. when $i\beta_0 = \omega_2/2,\ 3\omega_2/2$, the two ovals of the Cartesian come together into a circle with center at e_2 and having e_1 and e_3 as inverse points. And when u traces an eighth line, i. e. $i\beta_0 = \omega_2/4,\ 3\omega_2/4,\ 5\omega_2/4,\ 7\omega_2/4$, the map in the z-plane is a Cartesian which is the inverse of a Cassinian.

What has been said concerning the mapping of horizontal lines holds equally well for the mapping of vertical lines. The vertical lines of course map on the z-plane a system of confocal Cartesians orthogonal to those mapped by horizontal lines.

The Cartesian mapped by (2.1) is therefore expressed as a complex function of a real parameter α, since $u = \alpha + i\beta_0$. We can calculate its invariants by the method of Section 1.

Let us recall the fundamental relation between the p and p' functions, viz.

$$p'^2(u) = 4p^3(u) - g_2 p(u) - g_3,^* \tag{2.2}$$

from which it follows that

$$p''(u) = 6p^2(u) - g_2/2$$
$$p'''(u) = 12p(u)p'(u).$$

Following the customary notation in the theory of elliptic functions we shall denote by primes ($'$) differentiation with respect to the variable indicated. For example

$$p'u = d(pu)/du; \quad p'2u = d(p2u)/d(2u) = d(p2u)/2du.$$

We now have

$$\{z, u\} = z'''/z' - 3z''^2/2z'^2 = 12pu - 3p''^2u/2p'^2u$$

and by the addition formula for the p-function,

$$(p'u - p'v)^2/4(pu - pv)^2 = p(u+v) + pu + pv, \tag{2.3}$$

when $u = v$ we have the further fact that

$$p''^2u/4p'^2u = p2u + 2pu.$$

Making these simplifications the Schwarzian reduces to the compact form:

$$\{z, u\} = -6p2u.$$

Separating the Schwarzian into its real and imaginary parts each of which is a real function of u we have

$$A(u) = -3(p2u + p2\bar{u})$$
$$B(u) = 3i(p2u - p2\bar{u})$$
$$B'(u) = 6i(p'2u - p'2\bar{u})$$
$$B''(u) = 12i(p''2u - p''2\bar{u})$$

and
$$d\lambda = B^{\frac{1}{2}}du = B^{\frac{1}{2}}d(2u)/2$$

or
$$d(2u)/d\lambda = 2/B^{\frac{1}{2}}.$$

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* The coefficients g_2 and g_3 are invariants of the p-function; in terms of the e_i's they have the values: $g_2 = -4(e_2e_3 + e_3e_1 + e_1e_2)$; $g_3 = 4e_1e_2e_3$.

Making use of formula (1.2)

$$I_5 = \frac{i}{8\cdot 27(p2u - p2\bar{u})}\left[8\cdot 27(p2u + p2\bar{u}) + 144\,\frac{p''2u - p''2\bar{u}}{p2u - p2\bar{u}} - 180\left(\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}}\right)^2\right]$$

But from (2.2)

$$(p''2u - p''2\bar{u})/(p2u - p2\bar{u}) = 6(p2u + p2\bar{u})$$

and from (2.3)

$$(p'2u - p'2\bar{u})/(p2u - p2\bar{u}) = 4[p4\alpha + p2u + p2\bar{u}]$$

since $u + \bar{u} = 2\alpha$.

Therefore

$$I_5 = \frac{5i[p2u + p2\bar{u} - 2p4\alpha]}{3(p2u - p2\bar{u})}\,.$$

To obtain I_6 we must differentiate I_5 with respect to $2u$ and multiply by $2/B^{1/2}$. This gives

$$I_6 = \frac{k_1}{(p2u - p2\bar{u})^{3/2}}\left[p'2u + p'2\bar{u} - 4p'4\alpha + [2p4\alpha - p2u - p2\bar{u}]\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}}\right]$$

$$I_6 = \frac{2k_1}{(p2u - p2\bar{u})^{3/2}}\left[p4\alpha\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} - 2p'4\alpha + \frac{p2u\cdot p'2\bar{u} - p2\bar{u}\cdot p'2u}{p2u - p2\bar{u}}\right]$$

which can be arranged as follows:

$$I_6 = \frac{2k_1}{(p2u - p2\bar{u})^{3/2}}\left[(p4\alpha - p2\bar{u})\,\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} - 2p'4\alpha + p'2\bar{u}\right].$$

As a consequence of the addition formula (2.3) if $u_1 + u_2 + u_3 = 0$ we have the following relation:

$$\frac{p'u_2 - p'u_3}{pu_2 - pu_3} = \frac{p'u_3 - p'u_1}{pu_3 - pu_1} = \frac{p'u_1 - p'u_2}{pu_1 - pu_2}$$

which gives for $u_1 = 2u$, $u_2 = 2\bar{u}$, $u_3 = -4\alpha$ when we recall that p' is an odd function and p an even function:

$$\frac{p'2u - p'2\bar{u}}{p2u - p2\bar{u}} = \frac{p'2\bar{u} + p'4\alpha}{p2\bar{u} - p4\alpha}\,.$$

With this further substitution we now obtain:

$$I_6 = Kp'4\alpha/(p2u - p2\bar{u})^{3/2}.*$$

We see that $I_5 = \infty$ when $p2u = p2\bar{u}$. Let us write $u = \alpha + i\beta_0$ and then we will have $I_5 = \infty$

when $$p(2\alpha + 2i\beta_0) = p(2\alpha - 2i\beta_0)$$

or, since p is an even function,

when $$p(2\alpha + 2i\beta_0) = p(2i\beta_0 - 2\alpha).$$

This equality is satisfied when

$$2\alpha = n\omega_1, \qquad n, \text{ an integer.}$$

Hence $\alpha = n\omega_1/2$ and we need consider only those values of α which lie in the fundamental rectangle, i. e.

$$\alpha = 0,\ \omega_1/2,\ \omega_1,\ 3\omega_1/2.$$

For α equal to any of these values, the denominator of I_5 vanishes and the numerator becomes infinite,—hence $I_5 = \infty$.

Let us see what points on the Cartesian correspond to these values of α in the rectangle. We have previously mentiond the fact that the system of confocal curves orthogonal to those given by (2.1) with $u = \alpha + i\beta_0$ are those mapped by the equation

$$z = p(\alpha_0 + i\beta).$$

For $\alpha_0 = 0,\ \omega_1$, the map in the z-plane consists of the real axis and for $\alpha_0 = \omega_1/2,\ 3\omega_1/2$ the map in the z-plane is the circle with center e_1 and about e_3 and e_2. The eight points cut out of the Cartesian by this circle and real axis are therefore those points for which $I_5 = \infty$.

These eight points are special points of the Cartesian. To show this we proceed as follows. The parametric equation

$$z = p(\alpha + i\beta_0) \tag{2.4}$$

for real values of α maps a Cartesian in the z-plane as mentioned before. The conjugate point $\bar{z} = p(\alpha - i\beta_0)$ is also a point on the curve. For real α equation (2.4) gives points z of the curve.

Let us now allow α to take complex values. The map of

$$z = p(\alpha + i\beta_0)$$

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* K and k_1, are constant factors and $K = -6k_1$.

in the z-plane is no longer a point of the curve but it is a point which has associated with it an image point, y, in the Cartesian, y being given by the equation

$$\bar{y} = p(\alpha - i\beta_0).$$

Hence (2.4) and its conjugate give for real values of α points on the curve; for complex values of α, a point z and its image point y in the Cartesian.

A circle

$$z\bar{y} - c\bar{y} - \bar{c}z + c\bar{c} - r^2 = 0$$

and the biquadratic (2.4) have either common points of intersection or common inverse points. For the case in hand we suppose the circle and the biquadratic to intersect and we seek their common points in that sense. These points are points whose elliptic parameters, α, are zeros of

$$p(\alpha + i\beta_0) \cdot p(\alpha - i\beta_0) - cp(\alpha - i\beta_0) - \bar{c}p(\alpha + i\beta_0) + c\bar{c} - r^2.$$

This is an elliptic function of α having a double pole at $\alpha = i\beta_0$ and $\alpha = -i\beta_0$—hence it has four poles whose sum is zero and by Abel's Theorem which states that the sum of the zeros of an elliptic function must be congruent to the sum of the poles, it must have also four zeros, α_1, α_2, α_3, α_4; furthermore

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \mod 2\omega_1.$$

It is evident that at apses where a circle has four point contact with the biquadratic

$$4\alpha \equiv 0 \mod 2\omega_1$$

or

$$\alpha = n\omega_1/2 \qquad n, \text{ an integer.}$$

But these are the points of the biquadratic where $I_5 = \infty$. Therefore at apses on a biquadratic $I_5 = \infty$.

Returning to the hyperbola and parabola we see that the former has two apses at $\mu = \pm 1$ (the vertices) and the latter one apse at $\mu = 0$ (the vertex).

Let us now examine those points of the biquadratic where $I_6 = 0$. We shall find that they too are special points of the curve. At such points

$$p'4\alpha = 0,$$

that is

$$4\alpha \equiv \omega_1 \mod 2\omega_1$$

or

$$\alpha = \omega_1/4,\ 3\omega_1/4,\ 5\omega_1/4,\ 7\omega_1/4.$$

And, analogous to the preceding case, these points are cut from the biquadratic (2.4) by that curve of the confocal orthogonal system for which α_0 takes values congruent to $\omega_1/4$ mod $\omega_1/2$. This curve is mapped by the vertical lines which divide the period rectangle in eighths and as mentioned before it is the inverse of a Cassinian orthogonal to and having the same foci as the original biquadratic.

Proceeding as before suppose a biquadratic

$$a_{00}z^2\bar{z}^2 + 2a_{01}z^2\bar{z} + 2a_{10}z\bar{z}^2 + a_{02}z^2 + \cdots = 0$$

should intersect the given biquadratic (2.4). The elliptic parameters of these points of intersection are zeros of the elliptic function

$$a_{00}p^2(\alpha + i\beta_0)\cdot p^2(\alpha - i\beta_0) + 2a_{01}p^2(\alpha + i\beta_0)\cdot p(\alpha - i\beta_0) + \cdots.$$

Since this function has a four-fold pole at $\alpha = i\beta_0$ and $\alpha = -i\beta_0$ it must therefore have eight zeros

$$\alpha_1 + \alpha_2 + \cdots + \alpha_8 \equiv 0 \mod 2\omega_1.$$

In general only one biquadratic can be put through eight points; but if the eight points are the points of intersection of two biquadratics, through them a single infinity of biquadratics may be passed.

If now we have two biquadratics U_1, U_2 each of which has sixth order contact with a given curve C at P, then each member of the pencil $U_1 + kU_2$ will have seven points in common at P and an eighth point at, say, P'. Among this single infinity of biquadratics $U_1 + kU_2$, there will be one, U_0, having a double-point at P and one branch in fifth order contact with C and each member of the pencil; and furthermore U_0 will have necessarily the eighth point of intersection at P'. If now U_0 should have one branch in sixth order contact with any one of the pencil, P', the eighth point of intersection must coincide with P. Conversely, if the eighth point of intersection should coincide with P, then U_0 will have one branch in sixth order contact with each one of the pencil $U_1 + kU_2$.

Let us now take a biquadratic U given by

$$z = p(\alpha + i\beta_0).$$

Its points of intersection with a second biquadratic V will, in general, be eight in number. Let furthermore, the elliptic parameters of these eight points of intersection be α_i which must, we know, satisfy the condition

$$a_1 + a_2 + \cdots + a_8 \equiv 0 \mod 2\omega_1.$$

If now the biquadratic V is to have eight points in common with U at a point P, it must have a double-point at P and one of its branches must have sixth order contact with U at P; and if the elliptic parameter of the point P is α, then

$$8\alpha \equiv 0 \mod 2\omega_1.$$

We have seen that for $\alpha = \omega_1/4,\ 3\omega_1/4,\ 5\omega_1/4,\ 7\omega_1/4$ the differential invariant I_6 vanishes and at these points of U, V may have eight points in common with U, since for all of them $8\alpha \equiv 0 \mod 2\omega_1$. They are points cut out of U by the confocal inverted Cassinians mapped by the vertical lines of the rectangle.

These special points of the Cartesian we call Cassinian points or "apses of the second order." This latter terminology is suggested by the analogy with those points cut out by the map of the quarter lines of the rectangle where circles may have four point contact.

At these apses of the second order some loxodrome hyperosculates the biquadratic for there $I_6 = 0$. Furthermore all loxodromes consist entirely of points which are "apses of the second order."

3. *The Cornu Spiral.* As a final example of the differential inversive invariants of a curve we call attention to those curves whose ordinary intrinsic equation is of the type

$$r = as^m.$$

In this case the simplest method of finding I_5 is by making use of the relation

$$(d^2z/ds^2)/(dz/ds) = i/r = i/as^m.$$

Hence
$$\{z, s\} = d[(d^2z/ds^2)/(dz/ds)]/ds - (d^2z/ds^2)^2/2(dz/ds)^2$$
$$= -mi/as^{m+1} + 1/2a^2s^{2m},$$

giving
$$A = 1/2a^2s^{2m}; \quad B = -m/as^{m+1},$$

and now by formula (1.2)

$$I_5 = -1/2mas^{m-1} - a(m+1)(m-3)s^{m-1}/8m$$

and since

$$\lambda = (\pm m/a)^{\frac{1}{2}} \int s^{-(1+m)/2} ds = 2(\pm m)^{\frac{1}{2}} s^{(1-m)/2}/(1-m)a^{\frac{1}{2}} + \text{const.}$$

$$I_6 = (m-1)s^{(1-m)/2}/2m(\pm ma)^{\frac{1}{2}} - a^{3/2}(m^2-1)(m-3)s^{3(m-1)/2}/8(\pm m)^{3/2}$$

(3.1)

$$I_7 = -(m-1)^2/4m^2 - 3a^2(m-1)^2\ (m+1)(m-3)s^{2(m-1)}/16m^2$$

$$I_8 = -3a^{5/2}(m-1)^3(m+1)(m-3)s^{5(m-1)/2}/8m^2(\pm m)^{\frac{1}{2}}.$$

For $m = 1$ we have the loxodrome previously considered in Section 1. For $m = -1$ we have the ordinary intrinsic equation of the Cornu Spiral * or clothoid:

$$r = a/s.$$

The differential invariants of this curve are given by (3.1) for $m = -1$. They are

$$I_5 = s^2/2a = \lambda^2/2, \qquad \text{since } \lambda = \int ds/a^{\frac{1}{2}} = s/a^{\frac{1}{2}}$$
$$I_6 = s/a^{\frac{1}{2}} = \lambda$$
$$I_7 = 1$$
$$I_8 = 0.$$

We have here an example of a curve whose invariant I_7 is constant. The problem of finding all curves for which $I_7 = \text{const.}$ remains at yet unsolved. We have however another special case of curves of this type—namely those curves whose ordinary intrinsic equation is

$$r = As^3.$$

When $A = -1/a$ this is precisely the evolute of the Cornu Spiral, $r = a/s$. Referring again to (3.1) we see that for $m = 3$ the differential invariants of this evolute are

$$I_5 = -1/6As^2 = -\lambda^2/18$$
$$I_6 = 1/3^{3/2} A^{\frac{1}{2}} s = -\lambda/9$$
$$I_7 = -1/9$$
$$I_8 = 0.$$

* Loria, "Spezielle Algebraische und Transcendente Ebene Kurven," Art. 194.

We have given two particular examples of curves whose inversive intrinsic equation is

$$I_5 = a\lambda^2 + b\lambda + c$$

Professor Morley, in the reference given on p. 2, discusses the case of curves whose inversive intrinsic equation is

$$I_5 = a\lambda + b$$

and, as he points out in that paper, the general problem of finding those curves having the inversive intrinsic equation $I_5 = f(\lambda)$ involves the solution of the differential equation

$$\{z, \lambda\} = f(\lambda) \pm i.$$

Expansions of the Neumann Type in Terms of Products of Bessel Functions.

By Guy Stevenson.

I. Introduction.

1. *Statement of the Problem.* The object of this paper is the generalization of certain expansions of functions in terms of Bessel functions, and related functions, which expansions have been investigated by Neumann, Gegenbauer and others. In his *Theorie der Bessel'schen Funktionen* Neumann has obtained the expansion *

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z),$$

where $f(z)$ is single-valued and analytic in the neighborhood of the origin, and the coefficients a_n are constants whose values are reproduced in Art. 13. This expansion has been generalized by Gegenbauer † to one of the form ‡

$$z^{\nu} f(z) = \sum_{n=0}^{\infty} b_n J_{\nu+n}(z),$$

where ν is any complex quantity, except a negative integer, and the coefficients b_n are constants whose values are reproduced in Art. 13. Also, to Gegenbauer Watson attributes the expansion §

$$z^{\mu+\nu} f(z) = \sum_{n=0}^{\infty} c_n J_{\mu+n/2}(z)\, J_{\nu+n/2}(z),$$

where μ and ν are any complex quantities, except the halves of negative integers, and the coefficients c_n are constants whose values are also given in Art. 13.

In this paper we extend the expansion theories of Neumann and Gegenbauer in four general directions. In the first place each term in the generalized expansions contains, as factors, any number n of Bessel functions. In

* See also Watson, *Theory of Bessel Functions* (London, 1922), p. 523.

† Wiener Sitzungsberichte, 74 (2), (1877), pp. 124-130; Watson, *loc cit.*, p. 524.

‡ By an expression of the form a^r we shall uniformly mean $e^{r \log a}$ where the principal determination of $\log a$ is taken.

§ Wiener Sitzungsberichte, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

the second place the order of each of the Bessel functions in the expansions has the more general form $\mu_i + \theta_i m$ where the μ_i are constants, real or complex, the θ_i are certain positive constants and m is a non-negative integer. In the third place the argument of the Bessel functions is extended in an obvious manner, from z to $z - a$, where a is any real or complex quantity. And, finally, the extension is made to the expansion of functions of several variables.

We first obtain the expansion

$$(z/2)^{\alpha+k} = \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{F}_{k+2m}(z),$$

in which $\alpha = \mu_1 + \mu_2 + \cdots + \mu_n$, k is a non-negative integer, the coefficients $c_{k,m}$ are constants which are defined in Art. 2, and

$$\mathfrak{F}_m(z) = J_{\mu_1+\theta_1 m}(z) J_{\mu_2+\theta_2 m}(z) \cdots J_{\mu_n+\theta_n m}(z),$$

always subject to the following restrictions: none of the quantities $\mu_i + \theta_i m$ shall be a negative integer, for $i = 1, 2, \cdots, n$, and for $m = 0, 1, 2, \cdots$, and $\theta_1 + \theta_2 + \cdots + \theta_n$ shall be 1 or 2. Since the case presenting the most interest is that for which the sum of the θ's is 1 we make this assumption unless otherwise specified. We next develop an expansion of the function $f(z)$ which is analytic and single-valued within and upon the circle C defined by the relation $|z - a| = R$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m z^{-\alpha}\, \mathfrak{F}_m(z - a)$$

where

$$a_m = (1/2\pi i) \int_C B_m(t - a)\, f(t)\, dt,$$

and $B_m(t - a)$ is a polynomial in $1/(t - a)$ whose exact form is given in Art. 6. We then obtain expansions for functions in terms of the polynomials $B_m(z - a)$.

In the remaining sections a relation between these expansions and those of Taylor and Laurent is obtained, certain orthogonality properties are derived, some special cases are considered, expansions for the case for which the sum of the θ's is 2 are obtained, and, finally, the extension is made to the expansion of functions of several variables.

II. EXPANSION OF $(z/2)^{a+k}$ IN TERMS OF THE FUNCTIONS $\mathfrak{F}_{k+2m}(z)$.

2. *Expansion of* $\mathfrak{F}_m(z)$ *in powers of* $(z/2)$. If the absolutely convergent series which represent the Bessel functions $J_{\mu_i+\theta_i m}(z)$, namely,

$$(1) \qquad J_{\mu_i+\theta_i m}(z) = \sum_{r=0}^{\infty} (-1)^r (z/2)^{\mu_i+\theta_i m+2r}/r!\,\Gamma(\mu_i+\theta_i m+r+1), \qquad (i=1, 2, \cdots, n),$$

be multiplied together and the product arranged according to the powers of $z/2$ the coefficient of $(z/2)^{a+m+2s}$ is $(-1)^s a_{m,s}$, where

$$a_{m,s} = \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/i_1!\, i_2! \cdots i_n!\,\Gamma(\mu_1+\theta_1 m+i_1+1)\,\Gamma(\mu_2+\theta_2 m+i_2+1)\cdots\Gamma(\mu_n+\theta_n m+i_n+1),$$

the summation being taken over all non-negative integral solutions of the equation $i_1+i_2+\cdots+i_n=s$.

Since the series (1), which define the Bessel functions, all converge absolutely and uniformly in any closed region of the z-plane (the origin being excepted if the real part of $\mu_i+\theta_i m$ is negative) it follows that the expansion

$$(2) \qquad \mathfrak{F}_m(z) = \sum_{s=0}^{\infty} (-1)^s a_{m,s} (z/2)^{a+m+2s}$$

is also absolutely and uniformly convergent in the same region.

3. *Formal expansion of* $(z/2)^{a+k}$ *in terms of the functions* $\mathfrak{F}_{k+2m}(z)$. In order to determine the coefficients $c_{k,m}$ in the expansion

$$(3) \qquad (z/2)^{a+k} = \sum_{m=0}^{\infty} c_{k,m}\,\mathfrak{F}_{k+2m}(z), \qquad (k=0, 1, 2, 3, \cdots),$$

which expansion is momentarily assumed, we substitute for each of the functions $\mathfrak{F}_{k+2m}(z)$ the appropriate power series expansion obtained from (2) and equate coefficients of like powers of $(z/2)$. This gives the recurrence relations

$$(4) \qquad \begin{cases} a_{k,0}c_{k,0} & = 1, \\ a_{k,1}c_{k,0} - a_{k+2,0}c_{k,1} & = 0, \\ a_{k,2}c_{k,0} - a_{k+2,1}c_{k,1} + a_{k+4,0}c_{k,2} & = 0, \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ a_{k,m}c_{k,0} - a_{k+2,m-1}c_{k,1} + \cdots + (-1)^m a_{k+2m,0}c_{k,m} & = 0, \end{cases}$$

whence we have

$$(5)\quad c_{k,m} = \frac{\begin{vmatrix} a_{k,1} & a_{k+2,0} & 0 & 0 & \cdots & 0 \\ a_{k,2} & a_{k+2,1} & a_{k+4,0} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k,m-1} & a_{k+2,m-2} & a_{k+4,m-3} & \cdots & & a_{k+2m-2,0} \\ a_{k,m} & a_{k+2,m-1} & a_{k+4,m-2} & \cdots & & a_{k+2m-2,1} \end{vmatrix}}{a_{k,0}a_{k+2,0}\cdots a_{k+2m,0}}$$

$$= \frac{\begin{vmatrix} a'_{k,1} & 1 & 0 & 0 & \cdots & 0 \\ a'_{k,2} & a'_{k+2,1} & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{k,m-1} & a'_{k+2,m-2} & a'_{k+4,m-3} & \cdots & & 1 \\ a'_{k,m} & a'_{k+2,m-1} & a'_{k+4,m-2} & \cdots & & a'_{k+2m-2,1} \end{vmatrix}}{a_{k+2m,0}}$$

where

$$a'_{r,s} = a_{r,s}/a_{r,0} = \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/\, i_1!\, i_2! \cdots i_n!\, (\mu_1 + \theta_1 r + i_1)_{i_1} (\mu_2 + \theta_2 r + i_2)_{i_1} \cdots (\mu_n + \theta_n r + i_n)_{i_n}$$

and $$(a)_s = a(a-1)(a-2)\cdots(a-s+1).$$

If we denote by $\Delta_{k,m}(a')$ the determinant in the last member of (5) we have

$$(6)\quad c_{k,m} = \Delta_{k,m}(a')/a_{k+2m,0}.$$

In the next sections we show that the series

$$(7)\quad \sum_{m=0}^{\infty} c_{k,m}\,(z/2)^{-a}\,\mathfrak{F}_{k+2m}(z)$$

is absolutely and uniformly convergent in any closed region and that the series in (3) represents the function $(z/2)^{a+k}$.

4. *Expressions dominating* $a_{m,s}$, $c_{k,m}$ *and* $\mathfrak{F}_m(z)$. We first obtain simple dominants for the quantities $a_{m,s}$, $c_{k,m}$ and $\mathfrak{F}_m(z)$, which will be used later in the convergence proofs. From the definition of $a_{m,s}$ (Art. 2) we see that

$$(8)\quad |\, a_{m,s}\,| = \Big| \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/i_1!\, i_2! \cdots i_n!\, \Gamma(\mu_1+\theta_1 m+i_1+1) \times \Gamma(\mu_2+\theta_2 m+i_2+1)\cdots\Gamma(\mu_n+\theta_n m+i_n+1)\,\Big|$$

$$\leq \sum_{i_1, i_2, \ldots, i_n=0}^{s} \frac{s!/(i_1!\, i_2! \cdots i_n!\,)\;|\;(\mu_1+\theta_1 m+s)_{s-i_1}(\mu_2+\theta_2 m+s)_{s-i_2}\cdots(\mu_n+\theta_n m+s)_{s-i_n}\,|}{s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\,\Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

where the summation is taken over all non-negative integral solutions of the equation $i_1 + i_2 + \cdots + i_n = s$. If now we allow the summation to be taken over all non-negative integral values consistent with the relation $i_1 + i_2 + \cdots + i_n = s(n-1)$ then the last member of (8) will not be decreased.* That is, we have

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* The truth of this statement is established in the following manner:

By a comparison of the terms entering into the sum in the two cases for which $i_1 + i_2 + \cdots + i_n$ is s and $s(n-1)$ respectively, we observe that to each term in the first case there corresponds a term in the second case which differs from it only in the coefficients involving the factorials. Consequently, it is sufficient to show that, for each set of the i's consistent with the relation $i_1 + i_2 + \cdots + i_n = s$ we have

$$s!/i_1!\, i_2! \cdots i_n! \leq [s(n-1)]!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

This is true if

$$s!/[s(n-1)]! \leq i_1! i_2! \cdots i_n!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

Denote by $f(s, n;\ i_1, i_2, \cdots, i_n)$ the second member of the last inequality. The last inequality is obviously true for $n = 1$ or 2 if s is any non-negative integer. It is also clearly true when $s = 0$, 1, 2 for all values of n greater than 2. Therefore, in what follows, we may assume that s and n are each greater than 2. Let $p_1, p_2, p_3, \cdots, p_n$ denote a set of values which the i's may assume that will make the product $i_1!\, i_2! \cdots i_n!$ a minimum. Then we have

$$s!/i_1!\, i_2! \cdots i_n! \leq s!/p_1!\, p_2! \cdots p_n! \leq [s(n-1)]!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)!$$

for every possible set of the i's. Hence

$$s!/[s(n-1)]! \leq p_1!\, p_2! \cdots p_n!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)! = f(s, n;\ p_1, p_2, \cdots, p_n).$$

Now the proposition is established if it is shown that $f(s, n;\ p_1, p_2, \cdots, p_n)$ is no greater than $f(s, n;\ i_1, i_2, \cdots, i_n)$ for any set of values which the i's may take. If s and n are each greater than 2, and r is the integer satisfying the relation $r = s/n + \theta$ where $o \leqq \theta < 1$, then $f(s, n;\ i_1, i_2, \cdots, i_n)$ is seen to be simply related to $f(s, n;\ p_1, p_2, \cdots, p_n)$. In fact, after certain obvious reductions have been made, we have

$$f(s, n;\ i_1, i_2, \cdots, i_n) = f(s, n;\ p_1, p_2, \cdots, p_n)[(r+a_1)(s-r-a_1+1)] \cdots [(r+a_m)(s-r-a_m+1)]/[(r-b_1)(s-r+b_1+1)] \cdots [(r-b_m)(s-r+b_m+1)],$$

where a_i and b_i are suitably determined non-negative integers. Since no factor in brackets in the numerator of the fraction in the last relation can be less than $k(s-k+1)$ and no factor in brackets in the denominator can be greater than $k(s-k+1)$, it is clear that $f(s, n;\ i_1, i_2, \cdots, i_n) \geq f(s, n;\ p_1, p_2, \cdots, p_n)$ for every possible set of i's. Hence the proposition is established.

$$(9)\quad |\, a_{m,s}\,| \leq \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s(n-1)} \frac{[s(n-1)]\,!/i_1\,!\,i_2\,!\cdots i_n\,!\,|\,(\mu_1+\theta_1 m+s)_{i_1}(\mu_2+\theta_2 m+s)_{i_2}\cdots(\mu_n+\theta_n m+s)_{i_n}\,|}{s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\,\Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

$$= |\,\alpha+m+ns\,|_{s(n-1)}\,/s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|$$

$$\leq \Gamma(|\,\alpha\,|+m+ns+1)\,\Gamma(|\,\alpha\,|+m+s+1)\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|.$$

The summation indicated in the second member of (9) has been effected by means of the identity

$$(a_1+a_2+\cdots+a_n)_s = \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} [s!/i_1\,!\,i_2\,!\cdots i_n\,!]\,(a_1)_{i_1}(a_2)_{i_2}\cdots(a_n)_{i_n}$$

an identity which is readily established by mathematical induction. The special case for which $n=2$ was established by Vandermonde.* For convenience we shall denote by $\alpha_{m,s}$ the last member of (9).

From the recurrence relations obtained in Art. 2 we have the relation

$$(10)\quad |\,c_{k,m}\,| \leq [\,|\,a_{k,m}\,|\cdot|\,c_{k,0}\,|+|\,a_{k+2,m-1}\,|\cdot|\,c_{k,1}\,|+\cdots+|\,a_{k+2m-2,1}\,|\cdot|\,c_{k,m-1}\,|]/|\,a_{k+2m,0}\,|$$

$$\leq [\alpha_{k,m}\,|\,c_{k,0}\,|+\alpha_{k+2,m-1}\,|\,c_{k,1}\,|+\cdots+\alpha_{k+2m-2,1}\,|\,c_{k,m-1}\,|]/|\,a_{k+2m,0}\,|.$$

We shall denote the last member of (10) by $B_{k,m}$.

If we expand the determinant $\Delta_{k,m}(a')$ by minors with respect to the elements of the last column, and then expand in the same manner the minor of the unit element and so on, we have

$$(11)\quad \Delta_{k,m}(a') = a'_{k+2m-2,1}\Delta_{k,m-1}(a') - a'_{k+2m-4,2}\Delta_{k,m-2}(a') + \cdots + (-1)^m a'_{k+2,m-1}\Delta_{k,1}(a') + (-1)^{m+1}a'_{k,m}.$$

Now let $P_{k,m}(a')$ denote the sum of the absolute values of the terms in the expansion of $\Delta_{k,m}(a')$ as a polynomial in the symbols a'. Then

$$(12)\quad P_{k,m}(a') = |\,a'_{k,m}\,|+|\,a'_{k+2,m-1}\,|\,P_{k,1}(a')+|\,a'_{k+4,m-2}\,|\,P_{k,2}(a')+\cdots+|\,a'_{k+2m-2,1}\,|\,P_{k,m-1}(a').$$

If in (12) we replace $|\,a'_{r,s}\,|$ by the dominating quantity $\alpha'_{r,s}$, where

$$\alpha'_{r,s} = \Gamma(-|\,\alpha\,|+r+ns+1)/\Gamma(-|\,\alpha\,|+r+s+1)\,(-|\,\mu_1\,|+\theta_1 r+s)_s \times (-|\,\mu_2\,|+\theta_2 r+s)_s\cdots\ (-|\,\mu_n\,|+\theta_n r+s)_s,$$

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* See Chrystal's *Algebra*, Vol. 2, p. 9.

we have *

$$(13)\quad P_{k,m}(\alpha') = \alpha'_{k,m} + \alpha'_{k+2,m-1}P_{k,1}(\alpha') + \alpha'_{k+4,m-2}P_{k,2}(\alpha') + \cdots + \alpha'_{k+2m-2,1}P_{k,m-1}(\alpha').$$

Moreover, we have

$$(14)\quad |\, c_{k,m}\,| \leq P_{k,m}(\alpha')/|\, a_{k+2m,0}\,| \leq P_{k,m}(\alpha')/|\, a_{k+2m,0}\,|.$$

Making use of the inequality employed by Nielsen,†

$$|\, J_\nu(z)\,| \leq |\, (z/2)^\nu\,|\, e^{|z|^2/4|\nu_0+1|}/\Gamma(\nu+1),$$

where $|\, \nu_0+1\,|$ is the least of the quantities $|\, \nu+1\,|, |\, \nu+2\,|, \cdots$, we have the following inequality:

$$(15)\quad |\, \mathfrak{F}_m(z)\,| \leq |\, (z/2)^{\alpha+m}\,| \cdot |\, a_{m,0}\,|\, e^{|z/2|^2 E_m},$$

where $E_m = \sum_{i=1}^{n} 1/\nu_i(m)$, and $\nu_i(m)$ for each i is the smallest of the quantities $|\, \mu_i + \theta_i m + r\,|$ for $r = 1, 2, 3, \cdots$.

5. *Convergence of the series in* (3). In view of the inequalities (10) and (15) it is clear that the series in (3) is dominated, term by term, by the series

$$(16)\quad \sum_{n=0}^{\infty} B_{k,m}\,|\, a_{k+2m,0}\,| \cdot |\, (z/2)^{\alpha+k+2m}\,|\, e^{|z/2|^2 E_{k+2m}}.$$

Denote by U_m the mth term of (16), and consider the limit of the ratio U_{m+1}/U_m as m becomes infinite. From (16) and (10) we have

$$(17)\quad U_{m+1}/U_m = [B_{k,m+1}\,|\, a_{k+2m+2,0}\,|/B_{k,m}\,|\, a_{k+2m,0}\,|]\,|\, z/2\,|^2 \exp[\,|\, z/2\,|^2\{E_{k+2m+2} - E_{k+2m}\}]$$

$$\leq \left[\frac{\alpha_{k,m+1}\,|\, c_{k,0}\,| + \alpha_{k+2,m}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,2}\,|\, c_{k,m-1}\,|}{\alpha_{k,m}\,|\, c_{k,0}\,| + \alpha_{k+2,m-1}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,1}\,|\, c_{k,m-1}\,|} + \frac{\alpha_{k+2m,1}}{|\, a_{k+2m,0}\,|}\right] |\, z/2\,|^2 \exp[\,|\, z/2\,|^2\{E_{k+2m+2} - E_{k+2m}\}].$$

Consider the limit $\underset{m\to\infty}{L}\, f(m,s)$ where $f(m,s) = \alpha_{k+2s,\,m+1-s}/\alpha_{k+2s,\,m-s}$, in which s may take non-negative integral values subject to the restriction that it be not greater than m. From the definition of $\alpha_{r,s}$ (Art. 4) we see that

$$f(m,s) = \frac{[|\alpha|+k+2s+n(m-s)+1][|\alpha|+k+2s+n(m-s)+2]\cdots[|\alpha|+k+2s+n(m-s)+n]}{(m-s+1)[|\alpha|+k+s+m+1][\mu_1+\theta_1(k+2s)+m-s+1]\times[\mu_2+\theta_2(k+2s)+m-s+1]\cdots[\mu_n+\theta_n(k+2s)+m-s+1]}.$$

* We note that if $k = |\, u_i\,|/\theta_i$ for any i of the set $i = 1, 2, \cdots, n$, the quantities $P_{k,m}(\alpha')$ are undefined. Hence we exclude all such values of k.

† *Mathematischen Annalen*, Vol. 52, (1899) p. 231; see Watson, *loc. cit.*, p. 44.

We observe that the factors of the numerator may be paired with the last n factors of the denominator so that each of the resulting n ratios is bounded as to m and s. Let M denote the product of these n bounds. The limit $\underset{m\to\infty}{L} f(m,s)$ is then seen to be zero uniformly as to s when s is in the range $0 \leq s \leq m$.

Then, for every positive N, there exists an integer m' such that for all values of $m > m'$ and for all values of s in the range $0 \leq s \leq m$, we have

$$\alpha_{k+2s,\, m+1-s}\, N < \alpha_{k+2s,m-s}. \tag{18}$$

Then for all values of $m > m'$ we have

$$\frac{U_{m+1}}{U_m} \leq \frac{1}{N}\left[\frac{N\alpha_{k,m+1}\,|c_{k,0}| + N\alpha_{k+2,m}\,|c_{k,1}| + \cdots + N\alpha_{k+2m-2,2}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{N\alpha_{k+2m,1}}{|a_{k+2m,0}|}\right] R_m(z)$$

$$\leq \frac{1}{N}\left[\frac{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{\alpha_{k+2m,1}\,N}{|a_{k+2m,0}|}\right] R_m(z)$$

$$= \frac{1}{N}\left[1 + \frac{N\,\alpha_{k+2m,1}}{|a_{k-2m,0}|}\right] R_m(z),$$

where $R_m(z) = |z/2|^2 \exp\left[|z/2|^2 \{E_{k+2m+2} - E_{k+2m}\}\right]$,

and, since $\underset{m\to\infty}{L} \alpha_{k+2m,1}/|a_{k+2m,0}| = 0$, and $\underset{m\to\infty}{L} E_m = 0$, we see that

$$\underset{m\to\infty}{L}\, U_{m+1}/U_m \leq |z/2|^2/N < 1 \qquad \text{if} \qquad |z/2|^2 < N.$$

Hence the series (16) and consequently the series in (3) converges absolutely for all finite values of z.

The uniformity of convergence of the series (7) throughout any closed region is readily established. For, let R' be any constant, and, let $R > R'$. Then, if in (16) each term be divided by $(z/2)^a$ and z be replaced by R, we have a convergent series of positive constants by which the series in (7) is dominated term by term provided that $|z| \leq R'$. Hence the uniformity of convergence throughout the given region follows from the test of Weierstrass.

Since the power series expansions of the functions $(z/2)^{-a}\,\mathfrak{F}_{k+2m}(z)$ all converge absolutely for all finite values of z, and (7) converges uniformly in the circle $|z| = R'$, we may sum by columns the repeated series obtained by replacing each of the functions $(z/2)^{-a}\,\mathfrak{F}_{k+2m}(z)$ in (7) by its power series expansion,* and this sum is $(z/2)^k$, as is seen from the relations (4). Hence the series in (3) represents the function $(z/2)^{a+k}$.

* See Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, 4th Ed., p. 343.

III. Expansion of $f(z)$ in Terms of the Functions $F_m(z-a)$ and of the Functions $B_m(z-a)$.

6. *Expansion of $z^a/(t-z)$ in terms of $\mathfrak{F}_m(z)$.* In the series

$$z^a/(t-z) = \sum_{k=0}^{\infty} z^{a+k}/t^{k+1}, \tag{19}$$

which is valid when $|t| > |z|$, substitute the series expansions for z^{a+k} obtained from (3). This gives

$$z^a/(t-z) = \sum_{k=0}^{\infty} 2^{a+k}/t^{k+1} \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{F}_{k+2m}(z). \tag{20}$$

Rearrange this repeated series, assuming temporarily that this rearrangement is permissible, into the simple series

$$z^a/(t-z) = \sum_{m=0}^{\infty} B_m(t)\, \mathfrak{F}_m(z), \tag{21}$$

where

$$B_m(t) = \sum_{r=0}^{\leq m/2} c_{m-2r,r}\, 2^{a+m-2r}/t^{m+1-2r},$$

the sum terminating with $m/2$ if m is even, and with $(m-1)/2$ if m is odd.

Since the absolute convergence of a repeated series implies the absolute convergence of the corresponding double series,* the above rearrangement to obtain (21) is permissible provided the repeated series in (20) is absolutely convergent. To establish this fact we first observe that the repeated series in (20) is dominated, term by term, by the series

$$\sum_{k=0}^{\infty} |z^{a+k}|/|t|^{k+1} \sum_{m=0}^{\infty} P_{k,m}(a')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m}. \tag{22}$$

If we make use of the relations (12) we are able, by arguments entirely analogous to those employed in Art. 5, to prove that

$$\sum_{m=0}^{\infty} P_{k,m}(a')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m} \tag{23}$$

is convergent for every finite value of z and k. Also, by the aid of (13) we may prove in a similar manner that

$$\sum_{m=0}^{\infty} P_{k,m}(\alpha')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m} \tag{24}$$

is convergent for every finite value of z and for every $k > k_1$ where

* Bromwich, "Theory of Infinite Series," p. 78.

$k_1 > |\mu_i|/\theta_i$ for $i = 1, 2, \cdots, n$. (This restriction is placed upon k to insure that all the quantities $P_{k,m}(\alpha')$ shall exist.) For fixed $k \geq k_1$ we see from (14) that the series (23) is dominated, term by term, by the series (24). Denote by $S_k(z)$ the sum of (23), and by $\sigma_k(z)$ the sum of (24), when $k > k_1$. Now, if R is any constant and $|z| \leq R$, then $\sigma_k(z)$ is bounded both as to z and k. For, in the series (24) each term is a monotone decreasing function of k when $k \geq k_1$. Hence, if z and k are in the given regions, $\sigma_{k_1}(R) \geq \sigma_k(z)$. Clearly, if $0 \leq k < k_1$, $S_k(z)$ has a bound, since $S_k(z)$ exists for each value of k, and the number of S's is finite. Let this bound be M_1, and let M be the greater of the two quantities $\sigma_{k_1}(R)$ and M_1. Then we have

$$(25) \qquad \sum_{k=0}^{\infty} |2^{a+k}|/|t|^{k+1} \sum_{m=0}^{\infty} |c_{k,m}| \cdot |\mathfrak{F}_{k+2m}(z)| \leq M \sum_{k=0}^{\infty} |z^{a+k}|/|t|^{k+1},$$

which converges absolutely when $|z| < |t|$. Whence the rearrangement in obtaining (21) is justified.

The uniformity of convergence of the series $\sum_{m=0}^{\infty} B_m(t) z^{-a} \mathfrak{F}_m(z)$ with regard to both z and t throughout the regions $|t| \geq R$, $|z| \leq R'$, where $R > R'$, is readily established. For, when these inequalities are satisfied this series is dominated, term by term, by a series of positive quantities independent of t and z, and the uniformity of convergence follows from the test of Weierstrass. We may state this result as

THEOREM I. *Let z and t be two complex variables such that $|z| \leq R'$ and $|t| \geq R$, where $R > R'$; then the expansion*

$$(26) \qquad 1/(t-z) = \sum_{m=0}^{\infty} B_m(t) F_m(z),$$

where $F_m(z) = z^{-a}\mathfrak{F}_m(z)$, is valid. If t be a variable point on or outside the circle $|z| = R'$, and z a variable point on or within the circle $|z| = R'$, then the expansion converges uniformly with regard to t and z.

7. *Expansion of $z^a f(z)$ in terms of the functions $\mathfrak{F}_m(z)$.* Now, by the aid of Cauchy's integral formula and the expansion (26) we are able to obtain an expansion of a function which is analytic in the neighborhood of the origin in terms of the functions $F_m(z)$. For we have

$$(27) \qquad f(z) = (1/2\pi i) \int_C \frac{f(t)\,dt}{(t-z)}$$

$$= (1/2\pi i) \int_C \{ \sum_{m=0}^{\infty} B_m(t)\, F_m(z) \}\, f(t)\, dt$$

$$= \sum_{m=0}^{\infty} a_m F_m(z),$$

where

$$a_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, dt,$$

the path of integration being the circle $|z| = R$, within and upon which $f(z)$ is single-valued and analytic. This expansion converges uniformly with regard to z when $|z| \leq R'$ and $R > R'$. Hence the expansion

$$(28) \qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

is valid in the same region. We may state this result as

THEOREM II. *Let $f(z)$ be a function of z which is analytic and single-valued within and upon the circle C defined by the relation $|z| = R$; then the expansion*

$$(29) \qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

where

$$a_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, dt,$$

is valid when z is withn the circle $|z| = R$. If $R > R'$ the expansion obtained from (29) by dividing each term by z^a converges uniformly with regard to z when $|z| \leq R'$.

If we let $g(z) = 1/(t-z)$ and form from (21) $g(0), g'(0), g''(0), \cdots$, where the differentiation is with respect to z, it is clear that the polynominals $B_m(t)$ are uniquely determined.* And in a similar manner the expansion (27) is shown to be unique.

8. *Expansion of $f(z)$ in terms of $B_m(z)$.* Again if z and t be two complex variables which satisfy the relations $|z| \geq r$, $|t| \leq r'$, and $r > r'$;

* To be exact this proves that all expansions of $1/(t-z)$ in terms of the functions $F_m(z)$ which are differentiable, term by term, are identical. This is sufficient for our purpose. It can be proved, however, that *any* expansion of $1/(t-z)$ in terms of $F_m(z)$ which is convergent has a region of uniform convergence, and therefore is identical with (21).

then, by merely interchanging the roles of t and z in Art. 6 we obtain the expansion

$$1/(z-t) = \sum_{m=0}^{\infty} F_m(t)\, B_m(z). \tag{30}$$

If t be a variable point on or within the circle C defined by the relation $|t| = r'$, and z is any point on or outside the circle $|z| = r$, then the expansion (30) converges uniformly with regard to t and z. If $f(z)$ be any function of z which is analytic and single-valued upon and outside the circle C it can be shown, by arguments analogous to those used in Art. 7, to have the expansion

$$f(z) = \sum_{m=0}^{\infty} a'_m\, B_m(z), \quad \text{where} \quad a'_m = (1/2\pi i)\int_C F_m(t)\, f(t)\, dt. \tag{31}$$

And this expansion converges uniformly with regard to z provided z is on or outside the circle $|z| = r$.

9. *Expansion of $f(z)$ in terms of the functions $F_m(z-a)$ and $B_m(z-a)$.* If in the expansion (26) we replace t and z by $t-a$ and $z-a$, respectively, where a is any complex quantity, we obtain the expansion

$$1/(t-z) = \sum_{m=0}^{\infty} B_m(t-a)\, F_m(z-a), \tag{32}$$

which is valid when $|t-a| \geq R$, $|z-a| \leq R'$, and $R > R'$. This expansion converges uniformly with regard to t and z when these inequalities are satisfied.

In a similar manner we obtain from the expansion (30),

$$1/(z-t) = \sum_{m=0}^{\infty} F_m(t-a)\, B_m(z-a), \tag{33}$$

which is valid when $|t-a| \leq r'$, $|z-a| \geq r$, and $r > r'$. And likewise this expansion converges uniformly with respect to t and z when these inequalities hold.

If $f(z)$ is analytic and single-valued within and upon the circle C defined by the relation $|z-a| = R$, then the expansion

$$f(z) = \sum_{m=0}^{\infty} a_m\, F_m(z-a), \quad \text{where} \quad a_m = (1/2\pi i)\int_C B_m(t-a) f(t)\, dt, \tag{34}$$

is valid; and, if z remains in the circle $|z-a| = R'$, where $R > R'$, this expansion converges uniformly with respect to z.

And, if $f(z)$ is analytic and single-valued upon and outside the circle C', defined by the relation $|z-a|=r'$, then the expansion

$$(35)\quad f(z)=\sum_{m=0}^{\infty} a'_m B_m(z-a), \text{ where } a'_m=(1/2\pi i)\int_{C'} F_m(t-a)f(t)\,dt,$$

is valid; and, if z remains on or outside the circle $|z-a|=r$, where $r>r'$, this expansion converges uniformly with respect to z. When $a=0$ the expansions (34) and (35) reduce to (27) and (31), respectively.

IV. Relation to the Taylor and Laurent Expansions.

10. If the Taylor's expansion of $f(z)$ is $f(z)=\sum_{m=0}^{\infty} b_m(z-a)^m$, while the expansion (34) is

$$f(z)=\sum_{m=0}^{\infty} a_m F_m(z-a),$$

it is possible to obtain a relation between the coefficients b_m and a_m in these two expansions. For

$$(36)\qquad \begin{aligned} a_m &= (1/2\pi i)\int_C B_m(t-a)\,f(t)\,dt \\ &= (1/2\pi i)\int_C B_m(t-a)\,\{\sum_{s=0}^{\infty} b_s(t-a)^s\}\,dt \\ &= \sum_{s=0}^{\infty} b_s\,(1/2\pi i)\int_C (t-a)^s \sum_{r=0}^{\leq m/2} c_{m-2r,r}\,\frac{2^{a+m-2r}}{(t-a)^{m+1-2r}}\,dt \\ &= \sum_{r=0}^{\leq m/2} 2^{a+m-2r}\; c_{m-2r,r}\, b_{m-2r}. \end{aligned}$$

By means of these relations we are able to readily construct the expansion (29) when the Taylor's expansion for the given function $f(z)$ is known.

Next, let $f(z)$ be an analytic and single-valued function of z in the ring-shaped region S, bounded by the two circles C_2 and C_1 defined by the relations $|z-a|=r$ and $|z-a|=R$, respectively, where $R>r$. Let z be a fixed point in the region S. Then we have from Cauchy's integral formula

$$(37)\qquad f(z)=(1/2\pi i)\int_{C_1}\frac{f(t)\,dt}{t-z}+\frac{1}{2\pi i}\int_{C_2}\frac{f(t)\,dt}{t-z},$$

the two paths of integration being taken in a counter-clockwise direction. If we apply the expansions (32) and (33) to the first and second integrals, respectively, we have

$$(38)\qquad f(z) = (1/2\pi i)\int_{C_1} \sum_{m=0}^{\infty} B_m(t-a)F_m(z-a)f(t)\,dt + (1/2\pi i)\int_{C_2} \sum_{m=0}^{\infty} F_m(t-a)\,B_m(z-a)f(t)\,dt = \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt \quad \text{and} \quad a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

and this expansion is valid when z is in S. This result may be stated as

THEOREM III. *Let $f(z)$ be a function of z, which is analytic and single-valued in the region defined by the inequalities $0 < r \leq |z-a| \leq R$, and let C_1 and C_2 denote the circles $|z-a| = R$ and $|z-a| = r$, respectively. Then, the expansion*

$$(39)\qquad f(z) = \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt$$

and

$$a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

is valid and converges uniformly with regard to z when $r' \leq |z-a| \leq R'$, and $R > R' > r' > r > 0$.

If the Laurent expansion of $f(z)$ in the region $r' \leq |z-a| \leq R'$ is

$$(40)\qquad f(z) = \sum_{m=0}^{\infty} b_m(z-a)^m + \sum_{m=1}^{\infty} b'_m(z-a)^{-m},$$

we are able to obtain relations between the coefficients a_m, a'_m and the coefficients b_m, b'_m in the expansion (39). In fact the value of a_m is expressed in terms of b_m by a relation notationally equivalent to the relation (36) already obtained, and from (40) we have

$$(41)\qquad a'_m = (1/2\pi i)\int_C F_m(t-a)f(t)\,dt = (1/2\pi i)\int_C F_m(t-a)\left\{\sum_{s=1}^{\infty} b'_s(t-a)^{-s}\right\} dt = \sum_{s=1}^{\infty} b'_s\,(1/2\pi i)\int_C (t-a)^{-s} F_m(t-a)\,dt = \sum_{r=0}^{\infty} \frac{(-1)^r a_{m,r}}{2^{m+2r}}\; b'_{m+1+2r}.$$

By means of these relations it is possible to construct the Neumann series expansion of a given function when the Taylor or Laurent series for the given function is known.*

V. Orthogonality Properties.

11. We shall next obtain the following integral formulae:

$$\int_C F_m(t-a)\,F_r(t-a)\,dt = 0, \text{ for all values of } m \text{ and } r,$$

$$\int_C B_m(t-a)\,B_r(t-a)\,dt = 0, \text{ for all values of } m \text{ and } r,$$

(42)

$$\int_C F_m(t-a)\,B_m(t-a)\,dt = 2\pi i k, \text{ for } c \text{ enclosing the point } z=a,$$

$$= 0, \text{ for } c \text{ not enclosing the point } z=a,$$

$$\int_C F_m(t-a)\,B_r(t-a)\,dt = 0, \text{ for } m \neq r,$$

where the contour C, in each case, is any closed curve not passing through the point $z=a$. Denote by S the region enclosed by the contour C.

The first formula is true because the integrand is analytic throughout S and along its boundary, and hence Cauchy's theorem applies.

The second formula is seen to be true, by observing that the residue of $B_m(t-a)B_r(t-a)$ at $t=a$ is zero, and that it is analytic at all other points in S.

The third formula follows from the fact that the integrand has a pole of the first order at $t=a$ and at this point the residue is unity. We see that the integral is $2\pi i\kappa$ or 0 according as C does or does not enclose the point $z=a$ where κ is the excess of the number of positive over the number of negative circuits of C about the point $z=a$.

To establish the fourth formula we separate the argument into two cases, the one for which $m>r$ and the other for which $m<r$. If $m>r$ the residue of the integrand at $t=a$ is readily seen to be zero. Hence the formula is true for this case. If $m<r$ we may write $m=r-p$ where p is of the set $1, 2, 3, \cdots, r$. If p is odd the residue at $t=a$ is seen to be zero. If p is even denote it by $2s$. Then

* Cf. Neumann, *Theorie der Bessel'schen Funktionen* (Leipzig, 1867), pp. 33-35; also Watson, *l. c.*, pp. 523-524.

The residue of $F_m(t-a)B_{m+2s}(t-a)$ at $t=a$ is

$$a_{m,0}\,c_{m,s} - a_{m,1}\,c_{m+2,s-1} + \cdots + (-1)^s\,a_{m,s}\,c_{r,0},$$

which may be written in the form *

$$\frac{\begin{vmatrix} a_{m,0} & a_{m,0} & 0 & 0 & \cdots & 0 \\ a_{m,1} & a_{m,1} & a_{m+2,0} & 0 & \cdots & \\ a_{m,2} & a_{m,2} & a_{m+2,1} & a_{m+4,0} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{m,s-1} & a_{m,s-1} & a_{m+2,s-2} & a_{m+4,s-3} & \cdots & a_{m+2s-2,0} \\ a_{m,s} & a_{m,s} & a_{m+2,s-1} & a_{m+4,s-2} & \cdots & a_{m+2s-2,1} \end{vmatrix}}{\begin{vmatrix} a_{m,0} & a_{m+2,0} & \cdots & a_{m+2s,0} \end{vmatrix}}.$$

Since the first two columns of the determinant are identical the residue at $t=a$ is zero, and the fourth formula is verified. The formulae (42) afford a convenient means for determining the coefficients in some of the expansions which we have developed, provided the expansions are known to exist. For example, if we wish to determine the coefficients in the expansion (27) we assume an expansion of the form

$$f(z) = \sum_{m=0}^{\infty} a_m F_m(z),$$

multiply both members by $B_m(z)$, and integrate along the circle $|z| = R$, within and upon which $f(z)$ is assumed analytic and single-valued. Making use of the third and fourth formulae of (42) we obtain for a_m the value given in (27). It is clear that the coefficients in other of our expansions may be similarly determined.

VI. Expansions for which the Sum of the θ's is 2.

12. Thus far in our development we have assumed that the sum of the θ's in the functions $\mathfrak{F}_m(z)$ is unity. In fact that assumption was essential to the argument which was employed. For, if the sum of the θ's has *any* value other than unity, and k be not an even integer, we are led to the consideration of a system of inconsistent equations for determining the coefficients

* This statement is verified if we expand the determinant by minors with respect to the elements of the first column, and observe that the minor obtained from the determinant by striking out the first $(r+1)$ rows and the first $(r+1)$ columns is given by the relation

$$\Delta'_{m+2r,\,s-r} = c_{m+2r,\,s-r}\;a_{m+2r,\,0}\;a_{m+2r+2,\,0}\cdots a_{m+2s,\,0}.$$

in the formal expansion of $(z/2)^{a+k}$ in terms of the functions $\mathfrak{F}_m(z)$. However, if k be an even integer, and the sum of the θ's is 2, then we may obtain a system of equations which is consistent, and then it is possible to get an expansion for $(z/2)^{a+2r}$ in a series of the functions $\mathfrak{F}_{r+m}(z)$, in which the sum of the θ's is 2, and r is any non-negative integer. This suggests the possibility of the expansion of even analytic functions in terms of $z^{-a}\mathfrak{F}_m(z)$, in which the sum of the θ's is 2. Therefore, we next obtain such an expansion, then a similar expansion for odd functions, and finally by combining these two expansions get an expansion for any function which is analytic and single-valued in the neighborhood of the origin in terms of the functions $z^{-a}\mathfrak{F}_m(z)$ and $z^{-a+1}\mathfrak{F}_m(z)$ when the sum of the θ's is 2. In this article the quantities $a_{m,s}$ and $\mathfrak{F}_m(z)$ have precisely the same form as in the previous articles, but we shall understand that throughout this article we are carrying the assumption that the sum of the θ's is 2. The argument for this case is very similar to that used in the first part of this paper, and, consequently, much of the details may be omitted.

Assume an expansion of the form

$$(43) \qquad (z/2)^{a+2r} = \sum_{m=0}^{\infty} C_{r,m}\, \mathfrak{F}_{r+m}(z)$$

and substitute for each of the functions $\mathfrak{F}_{r+m}(z)$ the appropriate series expansion, and equate coefficients of like powers of $z/2$. From the resulting recurrence relations the quantities $C_{r,m}$ may be determined, and are seen to be

$$C_{r,m} = \frac{\begin{vmatrix} a_{r,1} & a_{r+1,0} & 0 & 0 & 0 \cdots 0 \\ a_{r,2} & a_{r+1,1} & a_{r+2,0} & 0 & 0 \cdots \\ a_{r,3} & a_{r+1,2} & a_{r+2,1} & a_{r+3,0} & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r,m-1} & a_{r+1,m-2} & a_{r+2,m-3} & a_{r+3,m-4} & \cdots a_{r+m-1,0} \\ a_{r,m} & a_{r+1,m-1} & a_{r+2,m-2} & a_{r+3,m-3} & \cdots a_{r+m-1,1} \end{vmatrix}}{a_{r,0} \quad a_{r+1,0} \quad \cdots \quad a_{r+m,0}}$$

The convergence of the series in (43) and its representation of the function $z/2^{a+2r}$ is proved by arguments analogous to those used in Arts. 2 to 5.

For each of the powers of $z/2$ in the expansion

$$z^a/(t^2 - z^2) = \sum_{r=0}^{\infty} (z^{a+2r}/t^{2r+2}),$$

substitute the corresponding expansion from (43) and rearrange the repeated series into a simple series. The resulting formal expansion,

$$(44) \qquad z^a/(t^2 - z^2) = \sum_{m=0}^{\infty} B_m(t)\, \mathfrak{F}_m(z),$$

where $$B_m(t) = \sum_{s=0}^{m} c_{s,m-s}\, (2^{a+2s}/t^{2s+2}),$$

is shown to be valid by the method used in Art. 6.

Now, let $f(z)$ be an even function of the complex variable z, single-valued and analytic, within and upon the circle C defined by the relation $|z| = R$. Then by the special form of Cauchy's integral formula employed by Neumann,* namely,

$$f(z) = (1/2\pi i) \int_C \frac{f(t)\, t\, dt}{(t^2 - z^2)},$$

and the expansion (44) we have

$$(45) \qquad z^a f(z) = (1/2\pi i) \int_C \frac{z^a f(t)\, t\, dt}{t^2 - z^2} = \sum_{m=0}^{\infty} c_m\, \mathfrak{F}_m(z),$$

where $$c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt.$$

This expansion is valid when z is inside the circle $|z| = R$, and converges uniformly with regard to z when $|z| \leq R'$, where $R > R'$. Since any odd function of z, which is single-valued and analytic at $z = 0$, may be expressed as z multiplied by an even analytic function, we are able to get the expansion for odd analytic functions in a series very similar to (45). For, let $\phi(z) = z f(z)$, where $\phi(z)$ is an odd analytic function of z. Then $f(z)$ is even, and has the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a}\, C_m\, \mathfrak{F}_m(z).$$

Consequently,

$$\phi(z) = \sum_{m=0}^{\infty} z^{-a+1}\, C_m\, \mathfrak{F}_m(z), \quad \text{where } C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt.$$

We may state these results as

THEOREM IV. *Let $f(z)$ and $\phi(z)$ be an even and an odd function of z, respectively, each analytic and single-valued in the region $|z| = R$, and along its contour C. Then the expansions*

$$(46) \qquad f(z) = \sum_{m=0}^{\infty} z^{-a}\, c_m\, \mathfrak{F}_m(z), \quad \textit{where } c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt,$$

$$(47) \qquad \phi(z) = \sum_{m=0}^{\infty} z^{-a+1}\, C_m \mathfrak{F}_m(z), \textit{ where } C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt,$$

* *Mathematische Annalen*, Vol. 3, p. 594.

are valid when z is within the circle $|z| = R$, and these expansions both converge uniformly with regard to z when $|z| \leq R'$, where $R > R'$.

Let $f(z)$ be any function of z which is analytic and single-valued in the region $|z| = R$ and upon its contour C. And, let $f(z) = \phi_1(z) + \phi_2(z)$, where $\phi_1(z)$ is an odd function, and $\phi_2(z)$ is an even function, each of which is analytic and single-valued in the circle $|z| = R$. Then by the expansions (46) and (47) we have at once the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a+1} c_m \mathfrak{F}_m(z) + \sum_{m=0}^{\infty} z^{-a} c'_m \mathfrak{F}_m(z), \tag{48}$$

where
$$c_m = (1/2\pi i) \int_C B_m(t)\, \phi_1(t)\, dt,$$

and
$$c'_m = (1/2\pi i) \int_C B_m(t)\, \phi_2(t)\, t\, dt.$$

This expansion is valid when z is within the circle $|z| = R$, and the convergence is uniform with regard to z when $|z| \leqq R'$, where $R > R'$. Furthermore, the uniqueness of the expansions (44) and (45) can be established in a manner analogous to that used in Art. 7.

VII. Special Cases.

13. In order to make explicit the sense in which these expansions are generalizations of known results it seems desirable to call attention to some of the special cases which have been treated. It will then become apparent what generalizations have been made.

When $n = 1$, $\mu_1 = 0$, and $\theta_1 = 1$ the expansion (29) reduces to that obtained by Neumann,* namely

$$f(z) = \sum_{m=0}^{\infty} a_m J_m(z),$$

where
$$a_m = (\epsilon_m/2\pi i) \int_C O_m(t)\, f(t)\, dt, \qquad \epsilon_m = 2 \text{ if } m \geq 1, \quad \epsilon_0 = 1,$$

and
$$\begin{cases} O_m(t) = (1/4) \sum_{r=0}^{\leq m/2} \dfrac{m(m-r-1)!}{r!\,(t/2)^{m+1-2r}} & \text{for } m \geq 1, \\ O_0(t) = 1/t. \end{cases}$$

For, since the expansion (21) is unique it follows that $\epsilon_m O_m(t) = B_m(t)$ when $\mu_1 = 0$ and $\theta_1 = 1$, n now having the value unity.

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* *Theorie der Bessel'schen Funktionen*, pp. 33-35; Watson, *loc. cit.*, p. 523.

When $n=2$, $\theta_1=\theta_2=1$, and $\mu_1=\mu_2=0$ we see from the uniqueness of the expansion (44) that the expansion of even functions given in (46) reduces to that obtained by Neumann * for even functions in terms of squares of Bessel Functions, namely,

$$f(z)=\sum_{m=0}^{\infty} a'_m\,[J_m(z)]^2, \quad \text{where } a'_m=(\epsilon_m/2\pi i)\int_C \Omega_m(t)\,f(t)\,t\,dt,$$

and
$$\begin{cases}\Omega_m(t)=(1/4)\sum_{s=0}^{m}\dfrac{m(m+s-1)\,!\,(s\,!)^2}{(m-s)\,!\,(2s)\,!\,(t/2)^{2s+2}} & \text{for } m\geq 1,\\ \Omega_0(t)=1/t^2.\end{cases}$$

And we have $\epsilon_m\Omega_m(t)=B_m(t)$, where $\theta_1=\theta_2=1$ and $\mu_1=\mu_2=0$, n now having the value 2.

When $n=1$, $\mu_1=\nu$, and $\theta_1=1$ the expansion (29) reduces to the following expansion, due to Gegenbauer,†

$$z^\nu f(z)=\sum_{m=0}^{\infty} a_m\,J_{\nu+m}(z), \text{ where } a_m=(1/2\pi i)\int_C A_{m,\nu}(t)\,f(t)\,dt,$$

and

$$A_{m,\nu}(t)=\frac{2^{\nu+m}(\nu+m)}{t^{m+1}}\sum_{s=0}^{\leq m/2}\frac{\Gamma(\nu+m-s)}{s\,!}(t/2)^{2s}.$$

From the uniqueness of the expansion (21) we have $A_{m,\nu}(t)=B_m(t)$ when $\mu_1=\nu$ and $\theta_1=1$, n now having the value 1.

When $n=2$, $\mu_1=\mu$, $\mu_2=\nu$ and $\theta_1=\theta_2=1/2$ we obtain from (29) another expansion, due also to Gegenbauer,‡

$$z^{\mu+\nu}f(z)=\sum_{m=0}^{\infty} a_m\,J_{\mu+(m/2)}(z)\,J_{\nu+(m/2)}(z),$$

$$a_m=(1/2\pi i)\int_C B_{m,\mu,\nu}(t)\,f(t)\,dt,$$

and
$$B_{m,\mu,\nu}(t)=\frac{2^{\mu+\nu+m}(\mu+\nu+m)}{t^{m+1}}$$

$$\sum_{s=0}^{\leq m/2}\frac{\Gamma[\mu+(m/2)-s+1]\,\Gamma[\nu+(m/2)-s+1]\,\Gamma(\mu+\nu+m-s)}{s\,!\,(\mu+\nu+m-2s+1)}\,(t/2)^{2s}.$$

And, likewise, $B_{m,\mu,\nu}(t)=B_m(t)$ when $\mu_1=\mu$, $\mu_2=\nu$ and $\theta_1=\theta_2=1/2$, n now having the value 2.

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* *Mathematische Annalen*, Vol. 3 (1871), p. 599; Watson, *loc. cit.*, p. 525.

† *Wiener Sitzungsberichte*, 74, (2), (1877), pp. 124-130; Watson, *loc. cit.*, p. 525.

‡ *Wiener Sitzungsberichte*, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

VIII. Expansions of Analytic Functions of Several Variables.

14. Much of the foregoing expansion theory which has been developed for functions of a single variable is capable of extension to functions of several variables. We shall next indicate briefly the argument for one of the expensions which can be obtained.

Since $\mathfrak{F}_m(z)$ is a function of the quantities μ_i and θ_i, we introduce the following notation:

$$\mathfrak{F}_m^{(j)}(z) = J_{\mu_{j1}+\theta_{j1}m}(z)\, J_{\mu_{j2}+\theta_{j2}m}(z) \cdots J_{\mu_{jv_j}+\theta_{jv_j}m}(z),$$

$$B_m^{(j)}(t) = \sum_{r=0}^{\leq m/2} c_{m-2r,r}\, \frac{2^{\alpha_j+m-2r}}{t^{m+1-2r}},$$

$$\alpha_j = \mu_{j1} + \mu_{j2} + \cdots + \mu_{jv_j},$$

with the restriction that $\mu_{ji} + \theta_{ji}r$ be not a negative integer for any i or j occurring in the expansion, and for $r = 0, 1, 2, \cdots$.

For simplicity in argument the expansion is developed for the case of two variables. The extension is obvious.

Let z_1, t_1, and z_2, t_2 be two pairs of complex quantities in the z_1 and z_2 planes, respectively, such that

$$(49)\quad \begin{cases} |z_1| \leq r_1, & |t_1| \geq R_1, & r_1 < R_1, \\ |z_2| \leq r_2, & |t_2| \geq R_2, & r_2 < R_2; \end{cases}$$

then, by Theorem I the expansions

$$(50)\quad \begin{cases} 1/(t_1 - z_1) = \sum_{m_1=0}^{\infty} B_{m_1}^{(1)}(t_1)\, F_{m_1}^{(1)}(z_1), \\ 1/(t_2 - z_2) = \sum_{m_2=0}^{\infty} B_{m_2}^{(2)}(t_2)\, F_{m_2}^{(2)}(z_2), \end{cases}$$

where $$F_{m_i}^{(r)}(z_i) = z^{-\alpha_r}\mathfrak{F}_{m_i}^{(r)}(z_i),$$

are valid. If t_1 and t_2 are variable points on or outside the circles C_1 and C_2, respectively, defined by the relations $|z_1| = R_1$, $|z_2| = R_2$, and if z_1 and z_2 are any variable points within or upon the circles $|z_1| = r$, and $|z_2| = r_2$, respectively, then the expansions (50) converge uniformly with regard to t_1, z_1 and t_2, z_2, respectively. Since both expansions converge uniformly when conditions (49) are fulfilled and are dominated by convergent series of constants, then under the same conditions the double series in the second member of the equation

$$(51)\qquad 1/(t_1 - z_1)\cdot 1/(t_2 - z_2) = \sum_{m_1,m_2=0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2)\, F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2)$$

converges uniformly.

Now let $f(z_1, z_2)$ be an arbitrary function of the two independent complex variables z_1 and z_2, which is analytic and single-valued in z_1 and z_2 separately when z_1 and z_2 are, respectively, in the regions bounded by the contours C_1 and C_2; then, by Cauchy's generalized integral formula, we have

$$\begin{aligned} f(z_1, z_2) &= (1/2\pi i)^2 \int_{C_1}\int_{C_2} \frac{f(t_1, t_2)\, dt_1\, dt_2}{(t_1 - z_1)(t_2 - z_2)} \\ &= (1/2\pi i)^2 \int_{C_1}\int_{C_2} \{\sum_{m_1,m_2=0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \\ &\qquad F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2)\}\, f(t_1, t_2)\, dt_1\, dt_2 \\ &= \sum_{m_1,m_2=0}^{\infty} A_{m_1,m_2} F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2), \end{aligned}$$

where $A_{m_1,m_2} = 1/(2\pi i)^2 \int_{C_1}\int_{C_2} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2)\, f(t_1, t_2)\, dt_1\, dt_2$.

This expansion is valid and converges uniformly with regard to z_1 and z_2 when z_1 and z_2 are, respectively, in the circles $|z_1| = r_1$ and $|z_2| = r_2$. This result may be stated for ν variables as

THEOREM V. *Let $f(z_1, z_2, \cdots, z_\nu)$ be a function of the complex quantities $z_1, z_2, \cdots, z_\nu$, which is analytic and single-valued when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the regions $|z_1| \leqq R_1$, $|z_2| \leqq R_2$, $\cdots$, $|z_\nu| \leqq R_\nu$, whose contours are, respectively, $C_1, C_2, \cdots, C_\nu$, then the expansion*

$$f(z_1, z_2, \cdots, z_\nu) = \sum_{m_1,m_2,\ldots,m_\nu=0}^{\infty} A_{m_1, m_2, \ldots m_\nu}\, F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2) \cdots F_{m_\nu}^{(\nu)}(z_\nu)$$

where

$$\begin{aligned} A_{m_1, m_2, \ldots m_\nu} = 1/(2\pi i)^\nu \int_{C_1}\int_{C_2}\cdots\int_{C_\nu} & B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \\ & \cdots B_{m_\nu}^{(\nu)}(t_\nu) f(t_1, t_2, \cdots, t_\nu)\, dt_1\, dt_2 \cdots dt_\nu, \end{aligned}$$

is valid and converges uniformly when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the circles $|z_1| = r_1$, $|z_2| = r_2$, $\cdots$, $|z_\nu| = r_\nu$, where $r_i < R_i$, $i = 1, 2, \cdots, \nu$.

It is now obvious that expansions of functions of several variables, analogous to the expansions (31), (34), (35) and (39) exist.

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

Canonical Forms for Ordinary Homogeneous Linear Differential Equations of the Second Order with Periodic Coefficients.

By Elijah Swift.

Even such a simple matter as the actual reduction to canonical form of the ordinary homogeneous linear differential equation of the second order,

$$L: \qquad y'' + p(x)y' + q(x)y = 0,$$

(p and q continuous in the interval (a, b), i. e. $a \leq x \leq b$) by transformations of the form

$$y = \lambda(x)\bar{y}, \qquad x = \phi(\bar{x}),$$

(where we assume that throughout (a, b) λ and ϕ are continuous together with their first two derivatives and $\lambda(x) \neq 0$, $\phi'(\bar{x}) \neq 0$) does not appear to have been carried out in the literature. However, in his lectures at Harvard University Professor G. D. Birkhoff has effected this very simply and for purposes of orientation, at his suggestion, I have given this material in the second section of this paper.

But there is another less simple case of great importance in the applications, namely, that in which $p(x)$ and $q(x)$ are periodic with period 2π. It is the problem of reducing a general equation of this type to a canonical form, preserving the periodic character of the coefficients, the solution of which is given in the following paper. Incidentally we classify such equations and investigate the general character of their solutions.

1. *Introductory.* In the following we consider the differential equation $L: \; y'' + p(x)y' + q(x)y = 0$, where the functions p and q are real and continuous in a closed interval (a, b), $a \leq x \leq b$, which may in particular include the whole of the X-axis. We consider only real values of x and real solutions of L. The following facts are well known: *

a. There exists one and only one function, $y(x)$, which is continuous together with its first two derivatives and satisfies the differential equation

* See e. g. Bocher, Leçons sur les Méthodes de Sturm, Chap. I for *a, b, d;* p. 46 for *c.*

identically throughout (a, b) and which is such that $y(x_0) = \alpha$, $y'(x_0) = \alpha'$, where x_0 is any arbitrarily chosen point of (a, b) and α and α' are arbitrarily chosen constants. From this follows that if a solution and its derivative vanish at the same point, then the solution vanishes identically throughout (a, b).

b. Two solutions, y_1, y_2, are said to be *linearly dependent* or simply *dependent,* if a relation of the form $C_1y_1 + C_2y_2 = 0$ (C_1, C_2, constants not both zero) holds identically throughout (a, b): linearly independent, if no such relation exists. If y_1, y_2 are two independent solutions, then any other solution may be written in the form $C_1y_1 + C_2y_2$ (C_1, C_2, suitably chosen constants).

c. If y_1, y_2 are two independent solutions, they cannot vanish at the same point and the zeros of y_1 and the zeros of y_2 occur alternately.

d. The "wronskian" of two solutions y_1, y_2, is defined as the determinant $y_1y'_2 - y'_1y_2$. Its value is

$$W(y_1, y_2) = K \exp\left[-\int p dx\right],$$

where K is a constant whose value depends upon which pair of solutions is chosen. If W vanishes at one point of (a, b) it vanishes identically throughout (a, b). Its vanishing is a necessary and sufficient condition that y_1 and y_2 be dependent.

With suitable restrictions the only transformations of the form

$$x = \phi(\bar{x}, \bar{y}), \quad y = \psi(\bar{x}, \bar{y}),$$

changing the original variables, x, y, to the new set, $\bar{x}, \bar{y}$, which carry *any* equation, L, into one of the same form are

$$A \qquad y = \lambda(x)\bar{y},$$

changing the dependent variable, and

$$B \qquad x = \phi(\bar{x}),$$

changing the independent variable.* I shall refer to these by the letters A and B respectively. We assume that both $\lambda(x)$ and $\phi(\bar{x})$ are continuous and possess continuous first and second derivatives throughout the interval under

* Although a proof of this fact is comparatively simple, I have been unable to find it in the standard texts. It may be noted that there exist *special* transformations carrying *special* equations of the form L into equations of the same form. Such a one e. g. is $y = \lambda(x)\bar{y} + y_1$ where y_1 is a solution of the original equation.

consideration,—(a, b) for λ: $(\bar{a}, \bar{b})$, the interval corresponding to (a, b) by the transformation B, for ϕ.

Applied formally A transforms the differential equation L into

$$\bar{y}'' + (p + 2\lambda'/\lambda)\bar{y}' + [(\lambda'' + p\lambda' + q\lambda)/\lambda]\bar{y} = 0. \tag{1}$$

If the coefficients of $\bar{y}'$ and $\bar{y}$ are to be continuous throughout (a, b), it is necessary and sufficient that $\lambda(x)$ be not zero in (a, b)* and this is also necessary and sufficient that the inverse of A shall exist throughout (a, b).

Similarly B formally transforms L into

$$y'' + [p(\phi)\phi' - \phi''/\phi']y' + [q(\phi)/\phi'^2]y = 0, \tag{2}$$

where $\phi = \phi(\bar{x})$ and the differentiations are performed with respect to $\bar{x}$. Here a necessary and sufficient condition for the continuity of the coefficients and the existence of a single valued inverse of the transformation is that $\phi'(\bar{x})$ shall not vanish for values of $\bar{x}$ in $(\bar{a}, \bar{b})$.

Transformations satisfying these conditions,—(λ, ϕ, continuous, possessing first and second derivatives, λ, ϕ' nowhere zero)—we term *proper*. It is clearly necessary to consider only *proper* transformations if our work is to be more than formal and if the coefficients of the transformed equation are to be continuous.

The proper A transformations form a group, since 1) the succession of two such is equivalent to a single one, 2) the identical transformation is one of the group, 3) the inverse of any proper A transformation exists and is a proper A. Similarly the proper B transformations form a group. Furthermore if a finite number of proper A and of proper B transformations be carried out in any order, they will be equivalent to a single proper A and a single proper B.

In the greater part of this paper we shall be concerned with equations whose coefficients are periodic with period 2π and shall consider only those (proper) transformations which leave the coefficients periodic (with the same period). For an A transformation this means that the coefficient of $\bar{y}'$, $p(x) + 2\lambda'/\lambda$, [see (1)] must be periodic and hence λ'/λ also. Consequently λ must be of the form $e^{cx}P(x)$, where c is a constant and $P(x)$ a periodic function which cannot vanish for any value of x in (a, b). Clearly

* If λ is a function of x, continuous throughout (a, b) and satisfying there the equation $\lambda'/\lambda = f(x)$, where $f(x)$ is continuous, λ cannot vanish. For a continuous solution of the differential equation $\lambda' - f(x)\lambda = 0$ either vanishes identically throughout (a, b) or not at all.

if λ is of this form both coefficients of the transformed equation will be periodic.

If the *same* B transformation is to carry *any* equation with periodic coefficients into one whose coefficients are periodic, it must in particular do this if p, q, are constants. Referring to the form of the transformed equation [see (2)] we see from the form of the coefficient of y that this is possible only if $\phi'(\bar{x}+2\pi)=\phi'(\bar{x})$,* or $\phi(\bar{x}+2\pi)=\phi(\bar{x})+C$, C a constant. For equations in which q has a minimum period of 2π, C must be a multiple of 2π † and we shall take it equal to 2π, since taking it equal to $2k\pi$ is equivalent to following a transformation where $C=2\pi$ by $\bar{x}=k\bar{\bar{x}}$. Conversely if ϕ is such that $\phi(\bar{x}+2\pi)=\phi(\bar{x})+2\pi$, the coefficients of the transformed equation will be periodic.

For lack of a better term we shall call transformations of the above types "*periodic*":—that is, a *periodic A transformation* is one for which

$$\lambda(x)=e^{cx}\cdot P(x),\quad P \text{ periodic and nowhere zero;}$$

a *periodic B transformation* is one for which

$$\phi(\bar{x}+2\pi)=\phi(\bar{x})+2\pi,\quad \phi' \text{ nowhere zero,}$$

which is clearly equivalent to

$$\phi(\bar{x})=\bar{x}+P(\bar{x}),\quad P \text{ periodic.}$$

In § 2 we shall reduce the general equation to a canonical form using general A and B transformations: in the following paragraphs we shall reduce the equation with periodic coefficients to one of a number of canonical forms using only periodic transformations.

2. *Reduction of the general Differential Equation to a Canonical Form.*‡ Suppose that we transform a given differential equation, L, by arbitrary A and B transformations. Since we have two functions, λ and ϕ, at our disposal and only two coefficients, p and q, appear in the original equation, it seems reasonable to assume that we can so determine λ and ϕ that the transformed equation will have coefficients which do not depend upon p and q, but which are the same for all differential equations or for all of a large class.

This has been effected formally somewhat as follows: Let y_1 and y_2 be

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* $\phi'(x+2\pi)=-\phi'(x)$ implies the vanishing of ϕ' contrary to hypothesis.

† $C=0$ implies the vanishing of ϕ'.

‡ The results contained in this section are due to Professor G. D. Birkhoff: see Introduction.

any two independent solutions of the given equation, L. Employ the transformations

$$y = y_2(x)\bar{y}. \tag{3}$$

$$\bar{x} = y_1(x)/y_2(x). \tag{4}$$

Since any solution of L may be written in the form $C_1y_1 + C_2y_2$, any solution of the transformed equation will have the form $C_1\bar{x} + C_2$, so that the transformed equation is simply $d^2\bar{y}/d\bar{x}^2 = 0$. Since, however, *we are studying the equation L in connection with an interval* (a, b), we must take care that the transformations employed are proper throughout this interval if our work is to be more than merely formal. But here the transformations (3) and (4) both cease to be proper if $y_2(x)$ vanishes. But if any solution vanishes twice in (a, b) every solution vanishes at least once (see § 1, c) and we can find no y_2 such that these transformations are proper throughout (a, b). In other words, while the above reduction is valid for any differential equation and an interval sufficiently short, it is not in general more than a purely formal reduction for a general equation and a general interval.

We may, however, reduce the equation L to a canonical form employing only transformations proper throughout (a, b). Let y_2 be a solution of L such that $y_2(a) = 0$ (see § 1, a). If $y_2(b)$ is not 0, choose the solution y_1 such that $y_1(b) = 0$: if $y_2(b) = 0$, let y_1 be any independent solution. We use the transformations

$$\arctan\,[y_2(x)/y_1(x)] = \bar{x}, \tag{5}$$

$$y_2/(y_1^2 + y_2^2)^{\frac{1}{2}} = \sin\bar{x}, \tag{6}$$

$$y = (y_1^2 + y_2^2)^{\frac{1}{2}} \cdot \bar{y}, \tag{7}$$

where of course (5) and (6) are essentially the same. With suitable conventions there is no discontinuity in the left-hand member of (5) at points where $y_1 = 0$: in particular we may use the equivalent equation

$$\operatorname{arccot}(y_1/y_2) = \bar{x}.$$

Thus $d\bar{x}/dx$ and $d^2\bar{x}/dx^2$ exist and are continuous even at such points. Since y_1 and y_2 cannot vanish together, it is clear that the coefficient of $\bar{y}$ in (7) cannot vanish at any point of (a, b). Moreover this coefficient possesses continuous first and second derivatives, since y_1 and y_2 do. Consequently (7) is a proper A transformation.

From (5) $d\bar{x}/dx$ is $W(y_1, y_2)/(y_1^2 + y_2^2)$, which exists and is different from 0 throughout (a, b) [see § 1, d]. Since we may take any constant

multiple of y_2, in particular $-y_2$, in place of y_2, we may assume this positive. The theorems on implicit functions show that we may solve equation (5), for x in terms of $\bar{x}$, that $dx/d\bar{x}$ will exist and be different from 0 and that $d^2x/d\bar{x}^2$ will exist and be continuous throughout (a, b). Thus (5) defines a proper B transformation.

From equations (5), (6), (7), we see that y_1 and y_2 are carried over by the above transformations into $\cos \bar{x}$ and $\sin \bar{x}$ respectively, so that the transformed equation is

$$d^2\bar{y}/d\bar{x}^2 + \bar{y} = 0.$$

To find the transformed interval, let x change continuously in (a, b) starting with the value a. We may take the value of $\bar{x}$ corresponding to a as 0. Then as x increases from a, $\bar{x}$ will increase from 0. When x has increased from a to the next following zero of y_2, (5) or (6) shows that $\bar{x}$ has increased from 0 to π, when x increases from this zero to the next, $\bar{x}$ increases from π to $2\pi, \cdots$. If y_2 has $k+1$ zeros in (a, b) (including a) and $y_2(b)$ is not zero, $\bar{x}$ will increase from 0 to a value between $k\pi$ and $(k+1)\pi$ when x increases from a to b. But since $y_1(b) = 0$, (5) tells us that this value is exactly $(k + 1/2)\pi$. If y_2 has $k+1$ zeros in (a, b) including a and b, this value is clearly $k\pi$.

We may state our results as follows: *Any linear differential equation of the second order, whose coefficients p and q are continuous functions of the independent variable, x, in an interval (a, b), $a \leq x \leq b$, may be transformed by proper transformations of the types $y = \lambda(x) \cdot \bar{y}$, $x = \phi(\bar{x})$, into the equation $d^2\bar{y}/dx^2 + \bar{y} = 0$, while the interval (a, b) is carried over into the interval*

$$0 \leq \bar{x} \leq (k + 1/2)\pi \text{ or } 0 \leq \bar{x} \leq k\pi,$$

the first for the case that that solution of the original equation which vanishes at a has exactly k other zeros in (a, b) but does not vanish at b: the second for the case that that solution which vanishes at a has exactly k other zeros in (a, b) of which b is one.

Suppose now we have a second equation, L', and a closed interval, (a', b'), in which its coefficients are continuous. Then a necessary and sufficient condition that L' be transformable into L by proper transformations, while the interval $(a', b',)$ is transformed into (a, b), is that that solution of L' which vanishes at a' has as many zeros in (a', b') as that solution of L which vanishes at a, has in (a, b) and that these solutions either both vanish at the other end of the respective intervals or both fail to vanish there.

That this condition is necessary is clear from the fact that the transformations considered do not change the number of zeros of a solution in a given interval nor its vanishing at the end of the interval: that it is sufficient follows from the fact that if satisfied, when we transform L and L' into the canonical equation, the canonical intervals will be the same. Since every transformation considered has an inverse, we can transform L' into L directly and (a', b') will go over into (a, b).

Thus if we consider the group of proper A and B transformations and a differential equation together with an interval in which its coefficients are continuous, essentially the only invariants are the number of zeros in the interval of a solution vanishing at one end and the existence or non-existence of a solution vanishing at both ends of the interval.

3. *Preliminary Study of the Differential Equation with Periodic Coefficients.** In what follows we shall deal exclusively with differential equations, L, whose coefficients, $p(x)$, $q(x)$, are continuous for all values of the real variable, x, and periodic with period 2π. For convenience I shall use the term periodic to mean periodic with period 2π.

If $y_1(x)$ and $y_2(x)$ are any two independent solutions of such an equation, $y_1(x + 2\pi)$ and $y_2(x + 2\pi)$ will also be solutions on account of the periodic character of the coefficients, and hence expressible in the form (see § 1, b)

$$(8) \qquad \begin{aligned} y_1(x + 2\pi) &= C_1 y_1(x) + C_2 y_2(x), \\ y_2(x + 2\pi) &= C'_1 y_1(x) + C'_2 y_2(x), \end{aligned} \qquad (C_1, C_2, C'_1, C'_2, \text{constants}).$$

Since we are dealing with real solutions, the C's will all be real. The value of their determinant may be found directly from the differential equation, L: it is †

$$(9) \qquad \Delta(C) = C_1 C'_2 - C_2 C'_1 = \exp\left[-\int_0^{2\pi} p(x)\,dx\right]$$

and is accordingly positive.

Although the value of $\Delta(C)$ [and, as we shall see below, $C_1 + C'_2$] is definitely determined when the differential equation is given, the values of the C's themselves depend upon the particular solutions, y_1, y_2, chosen. If, for instance, we had chosen αy_1 (α any constant) as one of the pair above in place of y_1, we should have found αC_2 instead of C_2, C_1 would have remained the same, etc. Below in tabular form we give the new values of the C's

* For most of the results of this paragraph see e. g. Goursat-Hedrick, *Mathematical Analysis*, Volume 2, Part 2, pp. 146 ff.

† Goursat-Hedrick, *loc. cit.*, p. 147.

corresponding to different pairs of solutions. α and β are arbitrary constants, not zero.

Replacing	we find the new values of the C's to be			
	of C_1	of C_2	of C'_1	of C'_2
y_1 by αy_1	C_1	αC_2	C'_1/α	C'_2
y_2 by βy_2	C_1	C_2/β	$\beta C'_1$	C'_2
y_1 by $y_1 + y_2$	$C_1 + C'_1$	$C'_2 + C_2 - C_1 - C'_1$	C'_1	$C'_2 - C'_1$

No one of these substitutions changes the value of $C_1 + C'_2$ or of $\Delta(C)$ and hence it may easily be shown that each of these quantities has the same value no matter which pair of solutions we choose in place of y_1, y_2.* The value of $\Delta(C)$ was given above: the value of $C_1 + C'_2$ cannot in general be found without integrating the differential equation.

In general the differential equation will have no periodic solution, but it may have one or more real solutions, $Y(x)$, not identically zero, which have the property † that

$$Y(x + 2\pi) = \rho Y(x), \ \rho \text{ a constant.} \tag{10}$$

I shall call such a solution a *principal solution.* We may find all principal solutions as follows. Any solution, in particular Y, may be written in the form $Ay_1 + By_2$, (A, B constants not both zero). Consequently, using equations (8)

$$\begin{aligned} Y(x + 2\pi) &= Ay_1(x + 2\pi) + By_2(x + 2\pi) \\ &= (AC_1 + BC'_1)y_1(x) + (AC_2 + BC'_2)y_2(x). \end{aligned} \tag{11}$$

But if Y is a principal solution,

$$Y(x + 2\pi) = \rho Y(x) = \rho A y_1(x) + \rho B y_2(x). \tag{12}$$

Comparing (11) and (12) and noting that y_1 and y_2 are independent (see § 1, b), we have

$$A(C_1 - \rho) + BC'_1 = 0, \qquad AC_2 + B(C'_2 - \rho) = 0. \tag{13}$$

Since A and B are not both zero, the determinant of their coefficients in equations (13) must vanish, that is

$$(C_1 - \rho)(C'_2 - \rho) - C'_1 C_2 = 0, \tag{14}$$

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* This may be proved directly: see e. g. Forsyth, *Theory of Differential Equations*, Part III, Vol. 4, pp. 405-6.

† Of course any constant multiple of Y has this same property. When we speak of two (distinct) principal solutions, we understand neither is a constant multiple of the other.

which we call the *characteristic equation.* It may be written in the form

$$\rho^2 - (C_1 + C'_2)\rho + C_1C'_2 - C_2C'_1 = 0, \tag{14'}$$

and hence is independent of the particular pair of solutions, y_1, y_2, chosen. We also note that the constant term is positive. See (9).

Conversely if ρ is a root of this characteristic equation, we can find A and B, not both zero, satisfying equations (13) and $Ay_1(x) + By_2(x)$ will be a principal solution. The value of ρ uniquely determines the ratio A/B and the corresponding principal solution (except for a constant factor) unless equations (13) are identically satisfied, that is unless $C_2 = 0$, $C'_1 = 0$, $C_1 = C'_2 = \rho$, in which case, as appears from equations (8), y_1 and y_2 are principal solutions (with the same multiplier, ρ) and every solution is a principal solution.

Since the characteristic equation is a quadratic in ρ with constant term positive, the following cases are possible:

I. Roots real and distinct, both positive or both negative.

II. Roots real and equal.

(a) Every solution a principal solution.

(b) One and only one principal solution.

III. Roots conjugate complex.

In the following paragraphs we classify differential equations by the roots of their characteristic equations as above and reduce those of each type to simple canonical forms employing only periodic transformations of types A and B.

4. *Roots of Characteristic Equation real and distinct.* Let the roots be ρ_1 and ρ_2 where we call ρ_2 that root for which $|\rho_2| \geq |\rho_1|$. Corresponding to these are two principal solutions, Y_1, Y_2 such that

$$Y_i(x + 2\pi) = \rho_i Y_i(x), \qquad (i = 1, 2).$$

Consider the expression

$$Y_i(x) \cdot \exp\left[-(x/2\pi)\log|\rho_i|\right].$$

When x is replaced by $x + 2\pi$, this preserves the same value (ρ_i positive) or changes sign (ρ_i negative). It is then periodic or semi-periodic * and we may write

* I use the latter term to denote a function, $f(x)$, such that $f(x + 2\pi) = -f(x)$.

$$(15) \qquad Y_1(x) = \exp\left[(x/2\pi)\log|\rho_1|\right]\cdot P_1(x),$$
$$Y_2(x) = \exp\left[(x/2\pi)\log|\rho_2|\right]\cdot P_2(x),$$

where P_1 and P_2 are either both periodic ($\rho_1, \rho_2 > 0$) or both semi-periodic ($\rho_1, \rho_2 < 0$) and both possess continuous first and second derivatives, since Y_1 and Y_2 do.

If the roots of the characteristic equation are equal but there are two independent principal solutions, we have essentially the above case and accordingly shall not rule out the possibility $\rho_1 = \rho_2$.

Y_1 and P_1 have the same zeros. If P_1 vanishes at x_1, it also vanishes at $x_1 + 2n\pi$, $(n = \pm 1, 2, \cdots)$, and since the zeros of Y_1 and Y_2 occur alternately, P_1 and P_2 vanish the same number of times in any interval of length 2π (including exactly one end point).

CASE (*i*). Y_1, Y_2 nowhere zero. If P_1 does not vanish in the interval $0 \leq x < 2\pi$, it can vanish nowhere on the X-axis and neither can P_2. Y_2/Y_1 is everywhere continuous and its derivative

$$(Y_1Y'_2 - Y_2Y'_1)/Y_1^2 = W(Y_1, Y_2)/Y_1^2$$

is nowhere zero, since Y_1 and Y_2 are independent solutions (see § 1, d). Incidentally we note that here we cannot have $\rho_1 = \rho_2$. For then Y_2/Y_1 would be a continuous periodic function (from (15)) whose derivative exists, is continuous and does not vanish for any value of x. But this is impossible, for a continuous periodic function cannot be monotonic.

We may state this result thus: *An ordinary linear differential equation of the second order with coefficients continuous in the interval* $0 \leq x \leq 2\pi$ *cannot have as independent solutions two functions of the form* $e^{cx}P_1(x)$, $e^{cx}P_2(x)$, *where* c *is a constant and* P_1 *and* P_2 *are continuous and periodic with period* 2π, *unless* P_1 (*and therefore* P_2 *also*) *vanishes at least once in the above interval.*

We now consider the transformation defined by the equation

$$(16) \qquad \log(Y_2/Y_1)^* = (x/2\pi)\log(\rho_2/\rho_1) + \log[P_2(x)/P_1(x)]$$
$$= (\bar{x}/2\pi)\log(\rho_2/\rho_1)$$

or, for convenience setting

$$(17) \qquad 2\alpha = (1/2\pi)\log(\rho_2/\rho_1).$$

$$(16') \qquad x + (1/2\alpha)\log(P_2/P_1) = \bar{x}.$$

The left-hand member of equation (16) possesses a continuous non-vanishing

* To avoid imaginaries we choose Y_1, Y_2 so that Y_2/Y_1 is positive.

derivative, namely $W(Y_1, Y_2)/Y_1Y_2$, and also a continuous second derivative (since Y_1 and Y_2 possess continuous second derivatives). From the theorems on implicit functions it follows that (16) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, so that ϕ' and ϕ'' exist and are continuous and that ϕ' is nowhere zero. Furthermore, on account of the periodicity of P_1 and P_2, when x increases by 2π, so will $\bar{x}$ and conversely. Thus equation (16) [or (16′)] defines a proper periodic B transformation.

Using equations (15), (16) and (17) we may write

$$(18) \qquad \begin{cases} Y_1 = \exp(-\alpha\bar{x}) \exp[(x/2\pi) \log (\rho_1\rho_2)^{\frac{1}{2}}] \cdot [P_1(x)P_2(x)]^{\frac{1}{2}}, \\ Y_2 = \exp(2\alpha\bar{x}) \cdot Y_1. \end{cases}$$

Consider now the A transformation

$$(19) \qquad y = \lambda(x) \cdot \bar{y}, \text{ where}$$
$$\lambda(x) = \exp[(x/2\pi) \log (\rho_1\rho_2)^{\frac{1}{2}}] \cdot [P_1(x)P_2(x)]^{\frac{1}{2}}.$$

λ possesses continuous first and second derivatives and is itself nowhere zero. It is also of the form given on p. 394 for a periodic transformation. It is therefore a proper, periodic A transformation.

From (18) we see that this transformation carries the solutions Y_1, Y_2 of the original equation over into the canonical forms $\exp(-\alpha\bar{x})$ and $\exp(+\alpha\bar{x})$, and the equation becomes

$$(20) \qquad d^2\bar{y}/d\bar{x}^2 - \alpha^2\bar{y} = 0.$$

Thus any equation may, in this case, be reduced to the canonical form (20) by proper periodic transformations, the only invariant being essentially α, which is determined by the ratio of the roots of the characteristic equation.

Case (*ii*). Y_2 vanishes for some value of x. Suppose that $Y_2(x)$ vanishes k times in the interval $0 \leq x < 2\pi$ where k is even if P_2 is periodic and odd if P_2 is semi-periodic. We assume $P_2(0) = 0$: if it is not, the transformation $x = \bar{x} + a$ will carry a zero, a, into the origin. We consider the B transformation defined by the equation

$$(21) \qquad Y_2/Y_1 = \tan(k/2)\bar{x} \cdot \exp(2\alpha\bar{x}) \cdot \exp[(-2\alpha^2/k^2) \cos k\bar{x}],$$

where α has the meaning assigned above by (17). In particular if $\rho_1 = \rho_2$, $\alpha = 0$.

The last (exponential) factor on the right of (21) is inserted to insure that the derivative of that member of the equation shall always be positive,—which is necessary for the validity of the transformation (see below). In fact the derivative of $\exp(2\alpha x) \cdot \tan(k/2)x$ is

$$[\exp(2\alpha x)/2\cos^2(k/2)x](2\alpha \sin kx + k),$$

which will vanish for some value of x if $k \leq 2\alpha$, while the derivative of the right-hand member of (21) is $(4\alpha^2 \sin^2 k\bar{x} + 4\alpha k \sin k\bar{x} + 2k^2)$ multiplied by a non-vanishing factor and can never vanish. The final factor in (21) is not uniquely determined: in particular instead of $2\alpha^2$ we might use $b\alpha^2$, $b > 1$. Of course if $k > 2\alpha$ we may omit this factor and have simpler canonical forms: those forms which I give, however, are valid for all values of k and α.

We now show that (21) defines a proper periodic transformation. For convenience denote the zeros of Y_2 by $0, a_1, a_2, \cdots, a_k(=2\pi)$ and those of Y_1 by $b_1, b_2, \cdots, b_k$. These occur in the order $0, b_1, a_1, b_2, a_2, \cdots, a_{k-1}, b_k, a_k = 2\pi$ (§ 1, c). The derivative of the left-hand member of (21) is $W(Y_1, Y_2)/Y_1^2$ and consequently has everywhere the same sign (which we may take as positive), except at the points b_i, where it fails to exist. The derivative of the right-hand member is always positive except at the points $[(2i-1)/k]\pi$, $(i = 1, 2, \cdots, k)$ where it fails to exist.

Since $Y_2(0) = 0$, the values $x = 0$ and $\bar{x} = 0$ satisfy (21) and we shall let these correspond to each other. Now consider the intervals

$$0 \leq x \leq b_1, \qquad 0 \leq \bar{x} \leq \pi/k.$$

When x increases from 0 to b_1, the left-hand member of (21) increases monotonically from 0 to ∞; and when $\bar{x}$ increases from 0 to π/k, the right-hand member likewise increases monotonically from 0 to ∞, so that (21) defines a continuous one-to-one correspondence between the points of the above intervals. Moreover in these intervals, except at the points $x = b_1$, $\bar{x} = \pi/k$, each member of (21) possesses continuous first and second derivatives and each first derivative is positive. It follows that $dx/d\bar{x}$ and $d^2x/d\bar{x}^2$ exist and are continuous, except perhaps at the upper ends of the intervals considered. But if we equate the inverse of one member of (21) to the inverse of the other, we see that here also the correspondence is continuous and the derivatives exist and are continuous.

We next consider the intervals $b_1 \leq x \leq b_2$, $\pi/k \leq \bar{x} \leq 3\pi/k$, for which similar considerations hold, each member of (21) increasing from $-\infty$ to $+\infty$ when x and $\bar{x}$ increase from the lower to the upper end of the respective intervals. Proceeding in this manner we find that for the intervals $0 \leq x \leq 2\pi$, $0 \leqq \bar{x} \leqq 2\pi$ equation (21) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, where ϕ is continuous and possesses continuous first and second derivatives and ϕ' is nowhere zero.

If in (21) we replace x by $x + 2\pi$, the left hand member is multiplied by ρ_2/ρ_1: similarly replacing $\bar{x}$ by $\bar{x} + 2\pi$ the right hand member is multiplied by $e^{4\pi\alpha}$ or ρ_2/ρ_1 (see (17)), so that if (21) is true for a pair of values, x, $\bar{x}$, it is also true for the pair, $x + 2\pi$, $\bar{x} + 2\pi$. It follows at once that we may obtain the correspondence between x and $\bar{x}$ for the interval $(2\pi, 4\pi)$ by taking corresponding values of x and $\bar{x}$ for the interval $(0, 2\pi)$ and increasing each by 2π: for the interval $(4\pi, 6\pi)$ from the interval $(2\pi, 4\pi)$ in the same manner etc.

Thus equation (21) defines x as a function of $\bar{x}$, $x = \phi(\bar{x})$, for all values of these variables, where ϕ' and ϕ'' exist and are continuous and ϕ' is nowhere zero, and $\phi(\bar{x} + 2\pi) = \phi(\bar{x}) + 2\pi$. Thus the transformation is a proper periodic transformation of type B.

To investigate the form into which the solutions of the original differential equation are carried by this transformation, we write

$$(22)\quad Y_1(\bar{x}) = \exp(-\alpha\bar{x}) \cdot \cos(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\cos^2(k/2)\bar{x}] \times \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot F(\bar{x}),$$

where $F(\bar{x})$ is a function whose properties are deduced below. From (21) follows

$$(22')\quad Y_2(\bar{x}) = \exp(+\alpha\bar{x}) \cdot \sin(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\sin^2(k/2)\bar{x}] \times \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)^{1/2}] \cdot F(\bar{x}).$$

From the manner of setting up the correspondence between x and $\bar{x}$, it is clear that the zeros of $Y_1(\bar{x})$ are the same as those of $\cos(k/2)\bar{x}$ and the zeros of $Y_2(\bar{x})$ are the same as those of $\sin(k/2)\bar{x}$. From (22) it appears that $F(\bar{x})$ is continuous and possesses continuous first and second derivatives everywhere except possibly at the zeros of $\cos(k/2)\bar{x}$. But (22′) shows that it is continuous and possesses continuous first and second derivatives at these points also. Furthermore F can vanish only at a common zero of Y_1 and Y_2 and is therefore nowhere zero. If in either (22) or (22′) we replace $\bar{x}$ by $\bar{x} + 2\pi$ and remember that Y_i is a principal solution (see p. 599), we verify that F is periodic.

The A transformation

$$(23)\qquad y = \lambda(\bar{x})\bar{y},\ \lambda = \exp[(\bar{x}/2\pi)\log(\rho_1\rho_2)]^{1/2} \cdot F(\bar{x})$$

is proper and periodic. It carries our solutions into the canonical forms

$$\exp(-\alpha\bar{x}) \cdot \cos(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\cos^2(k/2)\bar{x}],$$
$$\exp(+\alpha\bar{x}) \cdot \sin(k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2)\sin^2(k/2)\bar{x}].$$

The canonical form of the equation may be found from the solutions with some labor. It is

$$(24)\quad 4k^2[k^2+2\alpha k\sin k\bar{x}+2\alpha^2\sin^2 k\bar{x}]\bar{y}''-8k^3[\alpha k\cos k\bar{x}+\alpha^2\sin 2k\bar{x}]\bar{y}' + [(k^5-8\alpha^2k^4)-(2\alpha k^5+32\alpha^3k^3)\sin k\bar{x} -(36\alpha^4k^2+6\alpha^2k^4)\sin^2 k\bar{x}-24\alpha^5k\sin^3 k\bar{x}-8\alpha^6\sin^4 k\bar{x}]\bar{y}=0.$$

For the special case where every solution is a principal solution, $\rho_1=\rho_2$, $\alpha=0$ and the canonical forms for the solutions and the equation become

$$\sin(k/2)\bar{x},\ \cos(k/2)\bar{x}\ \text{and}\ \bar{y}''-(k^2/4)\bar{y}=0.$$

It thus appears that if the roots of the characteristic equation are real and distinct, the only invariants (under proper periodic transformations) are essentially (a) the ratio of these roots, (b) the number of zeros of a principal solution in the interval $0\leq x<2\pi$.

5. *Roots of Characteristic Equation equal.* If the roots of the characteristic equation are equal, there will be a principal solution, Y_1, such that $Y_1(x+2\pi)=\rho Y_1(x)$, where ρ is the root. If there is an independent solution, $Y_2(x)$, such that $Y_2(x+2\pi)=\rho Y_2(x)$, every solution will have this property. This case has been implicitly covered in the preceding paragraph (namely for $\alpha=0$) and we shall assume that it does not occur.*

Consider Y_1 and any independent solution, y_2. Equations (8) become for these

$$(25)\qquad Y_1(x+2\pi)=\rho Y_1(x),\qquad y_2(x+2\pi)=C'_1Y_1(x)+C'_2y_2(x).$$

From the form of the characteristic equation, the roots being equal, $C'_2=\rho$. C'_1 cannot be zero on account of our assumption above. By taking a suitable multiple of y_2, we can make $C'_1=2\pi\rho$ (see table, p. 598). Calling this multiple Y_2 we have

$$(26)\qquad Y_2(x+2\pi)=2\pi\rho Y_1(x)+\rho Y_2(x).$$

We readily verify from (25), (26) that $Y_1\cdot e^{-\beta x}$ and $[Y_2-xY_1]e^{-\beta x}$, where $\beta=(1/2\pi)\log|\rho|$ are periodic (semi-periodic if ρ is negative) so that we may write

$$(27)\qquad Y_1=e^{\beta x}P_1(x),\qquad Y_2=xY_1+e^{\beta x}P_2(x),$$

---

* This case and that mentioned below appear as distinct if we use elementary divisors in classifying the determinant, which, equated to zero, gives the characteristic equation. See Forsyth, *loc. cit.*

where P_1 and P_2 are periodic (semi-periodic) and possess continuous first and second derivatives, since Y_1, Y_2 do.

Although y_2 was any solution and Y_1 is not definitely determined because of a possible arbitrary constant factor, the wronskian of Y_1, Y_2, $W(Y_1, Y_2)$, has a definite sign. For choosing AY_1 in place of Y_1, Y_2 becomes AY_2 by (26) and $W(AY_1, AY_2) = A^2W(Y_1, Y_2)$ and has the same sign. Again any solution, Y^*, may be written in the form $C_1Y_1 + C_2Y_2$, where, if

$Y^* = xY_1 + e^{\beta x}P$, C_2 must be 1. But $W(Y_1, C_1Y_1 + Y_2) = W(Y_1, Y_2)$.

As in the preceding paragraph we consider two cases in reducing the differential equation to a canonical form: one in which Y_1 does not vanish for any value of x, the other in which Y_1 has at least one zero.

Case (i). $P_1(x) \neq 0$. If P_1 does not vanish for any value of x, the transformations

$$y = e^{\beta x}P_1(x)\bar{y}, \tag{28}$$

$$Y_2/Y_1 = x + (P_2/P_1) = \bar{x}, \tag{29}$$

may be easily shown to be proper and periodic. They carry the solutions into the canonical forms 1 and x and the differential equation into the canonical form $d^2\bar{y}/d\bar{x}^2 = 0$. It thus appears that for this case there are no invariants.

Case (ii). $P_1(x)$ vanishes. Now let Y_1 vanish k times in the interval $0 \leq x < 2\pi$, where, as before, if ρ is positive, k will be even and P_1 periodic: if ρ is negative, k will be odd and P_1 semi-periodic. If $Y_1(0)$ is not 0, we can choose Y_2 so that $Y_2(0) = 0$: if $Y_1(0) = 0$, a linear transformation, $x = \bar{x} + a$, will reduce this to the above.*

We employ the transformation defined by the equation

$$Y_2/Y_1 = x + (P_2/P_1) = \bar{x} + \mu \tan (k/2)\bar{x} \tag{31}$$

where $\mu = 1$, $W(Y_1, Y_2) > 0$; $\mu < 0$ and $-\mu > 2/k$, $W(Y_1, Y_2) < 0$.† Essentially the same reasoning as used for the transformation defined by (21), with a slight change if $W < 0$, shows that (31) defines a proper periodic B transformation and employing the same methods as on pages 603

* The transformation $Y_2/Y_1 = \bar{x} - \mu \cot (k/2)\bar{x}$ may be used in this case to reduce the equation to a simple canonical form. Its effect is to interchange $\sin (k/2)\bar{x}$ and $\cos (k/2)\bar{x}$ in the canonical forms of the solutions, and change the sign of μ.

† If $W < 0$, the derivative of the left hand member of (31) is negative and if x and $\bar{x}$ are to increase together, the derivative of the right hand member must be negative also. The single value, $\mu = -3$, will clearly satisfy both inequalities since $k \geq 1$.

and 604, we see that (31) together with a proper periodic A transformation will reduce our solutions to the canonical forms

$$\cos (k/2)\bar{x} \quad \text{and} \quad \bar{x}\cos (k/2)\bar{x} + \mu \sin (k/2)\bar{x}. \tag{32}$$

The canonical form of the equation is

$$4\bar{y}''[\cos k\bar{x} + 1 + \mu k] + 4k\bar{y}' \sin k\bar{x} + \bar{y}[3k^2 + \mu k^3 - k^2 \cos k\bar{x}] = 0. \tag{33}$$

If the roots of the characteristic equation are equal, the only invariants are essentially (a) k, the number of zeros of the principal solution in the interval $0 \leq x < 2\pi$, (b) the sign of the wronskian, $W(Y_1, Y_2)$, where Y_1 is the principal solution and Y_2 is written in the form $Y_2 = xY_1 + e^{\beta x}P_1(x)$.

6. *Roots of Characteristic Equation Complex.* Let the absolute value of each of the roots be ρ and their angles be ϕ, $2\pi - \phi$, where we may take ϕ as lying between 0 and 2π, leaving undecided for the moment whether we shall call ϕ the larger or the smaller angle. If ρ is not 1, the proper periodic transformation

$$y = \exp [(x/2\pi) \log \rho] \cdot \bar{y}$$

will reduce the equation to one for which $\rho = 1$. We assume this carried out so that the roots are of the form $\cos \phi \pm i \sin \phi$.

For any pair of independent solutions, y_2, y_1, as we saw in § 3, the characteristic equation has the form

$$(C_1 - \rho)(C'_2 - \rho) - C'_1C_2 = 0. \tag{14}$$

Noting that C'_1C_2 is not 0 (since the C's are real and the roots of (14) imaginary) we see, referring to the table of § 3, that we can find two independent solutions, Y_1, Y_2, readily determined from y_1, y_2 and the constants, $C_1, \cdots, C'_2$, which have the property that the (new) constants, $C_1, \cdots, C'_2$, associated with Y_1, Y_2 by equations (8) are such that $C_1 = C'_2$, $C'_1 = \pm C_2$, and C'_1 has the same sign as $\sin \phi$. Since the roots of (14) are $\cos \phi \pm i \sin \phi$, we must then have

$$C_1 = C'_2 = \cos \phi, \quad C'_1 = -C_2 = \sin \phi.$$

We have thus found a pair of solutions, Y_1, Y_2, such that (substituting the values for the C's in (8))

$$\begin{cases} Y_1(x + 2\pi) = \cos \phi\, Y_1(x) - \sin \phi\, Y_2(x), \\ Y_2(x + 2\pi) = \sin \phi\, Y_1(x) + \cos \phi\, Y_2(x). \end{cases} \tag{34}$$

I shall call this pair, *written in this order,* a pair of principal solutions.

Equations (34) may be written in the form

$$Y_2(x+2\pi) = \cos(2\pi-\phi)\, Y_2(x) - \sin(2\pi-\phi)\, Y_2(x),$$
$$Y_1(x+2\pi) = \sin(2\pi-\phi)\, Y_2(x) + \cos(2\pi-\phi)\, Y_1(x),$$

which are of the same form as (34) except that $2\pi-\phi$ appears in place of ϕ and Y_1 and Y_2 are interchanged. It follows that if Y_1, Y_2 are a pair of principal solutions for one of the two possible determinations of the angle, ϕ, Y_2, Y_1 will be such a pair for the other. Since $W(Y_1, Y_2) = -W(Y_2, Y_1)$, we can choose for ϕ that one of the two possible values, ϕ, $2\pi-\phi$, for which the wronskian of a pair of principal solutions is positive. We shall assume this done at the outset, so that $W(Y_1, Y_2) > 0$. That this choice does not depend upon the particular pair of principal solutions chosen will appear below.

Consider the functions, $P_1(x)$, $P_2(x)$, defined by the equations

$$(35) \qquad \begin{cases} P_1(x) = \cos[(x/2\pi)\phi]\, Y_1(x) + \sin[(x/2\pi)\phi]\, Y_2(x), \\ P_2(x) = -\sin[x/2\pi)\phi]\, Y_1(x) + \cos[(x/2\pi)\phi]\, Y_2(x). \end{cases}$$

They are clearly continuous and possess continuous first and second derivatives: if we substitute $x+2\pi$ for x and reduce, using equations (34), we see that they are periodic. We may solve equations (35) and express Y_1, Y_2 in terms of P_1, P_2, thus

$$(36) \qquad \begin{cases} Y_1(x) = \cos[(x/2\pi)\phi]\, P_1(x) - \sin[(x/2\pi)\phi]\, P_2(x), \\ Y_2(x) = \sin[(x/2\pi)\phi]\, P_1(x) + \cos[(x/2\pi)\phi]\, P_2(x). \end{cases}$$

Since Y_1, Y_2 are independent solutions, they cannot vanish together and hence neither can P_1, P_2. Thus any pair of principal solutions may be written in the form (36) and conversely we may readily verify that any pair of solutions written in this form will satisfy equations (34) and be a pair of principal solutions.

Equations (36) show that we may obtain Y_1, Y_2 from P_1, P_2 by rotating the P_1, P_2 plane about the origin through the angle $(x/2\pi)\phi$. It is clear that if we rotate this whole figure through any constant angle, we shall get a new set of functions, Y_3, Y_4, P_3, P_4 in the same relation to each other as Y_1, Y_2, P_1, P_2 and should expect Y_3, Y_4 to constitute a pair of principal solutions. In fact we readily verify that Y_3, Y_4 will be a pair of principal solutions *if and only if* they are expressible in the form

$$(37) \qquad Y_3 = A_1Y_1 - A_2Y_2, \quad Y_4 = A_2Y_1 + A_1Y_2, \qquad A_1, A_2 \text{ constants not both } 0,$$

corresponding to a rotation of Y_1, Y_2 through the angle arctan A_2/A_1 (and multiplication by a constant factor). From (37) follows

$$W(Y_3, Y_4) = (A_1^2 + A_2^2)\, W(Y_1, Y_2),$$

so that the sign of the wronskian of any pair of principal solutions will be the same as the sign of the wronskian of Y_1, Y_2, thus showing that the determination of ϕ adopted above did not depend upon the particular pair of principal solutions chosen.

From (36) we have

$$\text{(38)} \qquad \arctan\,(Y_2/Y_1) = (\phi/2\pi)x + \arctan\,(P_2/P_1).^*$$

Since P_1 and P_2 are continuous and do not vanish together, $\arctan\,(P_2/P_1)$ is continuous. When x increases continuously from 0 to 2π, $\arctan\,(P_2/P_1)$ will, on account of the periodicity of P_1 and P_2, change by an integral multiple of 2π, say $2k\pi$. Hence $\arctan\,(Y_2/Y_1)$, which is also everywhere continuous, changes by $\phi + 2k\pi$. Since the derivative of $\arctan\,(Y_2/Y_1) = W(Y_1, Y_2)/(Y_1^2 + Y_2^2)$ is positive, $\arctan\,(Y_2/Y_1)$ must actually increase and k is 0 or positive.

By a suitable choice of A_1, A_2 in (37), we can find a pair of principal solutions, uniquely determined except for a constant factor, such that the second vanishes for $x = 0$. We shall assume this to be the pair Y_1, Y_2.

We employ the transformation defined by the equations

$$\text{(39)} \quad \arctan\,(Y_2/Y_1) = (\phi/2\pi)x + \arctan\,(P_2/P_1) = (\phi/2\pi)\bar{x} + k\bar{x},$$

or the equivalent

$$\text{(39′)} \qquad Y_2/Y_1 = \tan\,[(\phi/2\pi) + k]\bar{x}.$$

Using reasoning similar to that previously employed (§ 4), we see that this defines a proper periodic B transformation.

Rewriting (39′) in the equivalent forms

$$\text{(40)} \qquad \begin{aligned} Y_2/(Y_1^2 + Y_2^2)^{\frac{1}{2}} &= \sin\,[(\phi/2\pi) + k]\bar{x}, \\ Y_1/(Y_1^2 + Y_2^2)^{\frac{1}{2}} &= \cos\,[\phi/2\pi) + k]\bar{x}, \end{aligned}$$

we note that as $\bar{x}$ increases from 0 to 2π, $[(\phi/2\pi) + k]\bar{x}$ increases from 0 to $2k\pi + \phi$ and will consequently take on at least $2k + 1$ values for which Y_2 vanishes, namely $0, \pi, 2\pi, \cdots, 2k\pi$. If $0 < \phi < \pi$, there will be no other value for which Y_2 vanishes, but if $\pi < \phi < 2\pi$, there will be one other such value, namely $(2k + 1)\pi$. Since the intervals $0 \leq x \leq 2\pi$ and $0 \leq \bar{x} \leq 2\pi$ correspond in a one-to-one manner and continuously by virtue of (39), Y_2 has the same number of zeros in the interval $(0, 2\pi)$ whether we regard x or $\bar{x}$ as the independent variable. Furthermore any solution which vanishes at 0

* Assuming all inverse tangents suitably defined at points where the denominator vanishes.

is a constant multiple of Y_2. It follows that if a solution (not identically zero) which vanishes for $x = 0$ has n zeros (including 0) in the interval $(0, 2\pi)$, then if n is odd, $2k + 1 = n$ and ϕ lies between 0 and π, while if n is even, $2k + 2 = n$ and ϕ lies between π and 2π. Thus this number of zeros determines the value of k and also the half plane in which ϕ lies.

From (36) we have $Y_1^2 + Y_2^2 = P_1^2 + P_2^2$. Hence the A transformation

$$y = (Y_1^2 + Y_2^2)^{1/2}\,\bar{y} \tag{41}$$

is proper and periodic. Together with (39) it carries the solutions, Y_1, Y_2, into the canonical forms [note (40)]

$$\cos[(\phi/2\pi) + k]\bar{x} \quad \text{and} \quad \sin[(\phi/2\pi) + k]\bar{x}. \tag{42}$$

The corresponding canonical form for the equation is

$$\bar{y}'' + [(\phi/2\pi) + k]^2\bar{y} = 0. \tag{43}$$

Thus if the roots of the characteristic equation are complex, the only invariants are (a) the angle (but not the absolute value) of the roots, (b) the number of zeros in the interval $(0, 2\pi)$ of a solution which vanishes at 0.

7. *Impossibility of further Reduction.* The proper periodic A transformation, $y = e^{cx}P(x) \cdot \bar{y}$, c a real constant, P nowhere zero, divides the roots of the characteristic equation by $e^{2\pi c}$, but does not change the character of the roots nor their ratio nor their angle, if they are complex. Clearly also it does not change the zeros of a solution and it carries a principal solution into a principal solution.

Again the proper periodic B transformation, $x = \phi(\bar{x})$, ϕ' nowhere zero, since $x + 2\pi = \phi(\bar{x} + 2\pi)$, leaves unchanged the characteristic equation and the number of zeros of any solution in an interval of length 2π. It also transforms a principal solution into a principal solution.

Examining the canonical forms and invariants as listed in the Summary, § 8, it appears at once from the above that the classes I (i), I (ii), II a, II b (i), II b(ii), III are actually distinct, in that no equation of one class may be transformed into one of another class by proper periodic A and B transformation and that the numbers α, k and ϕ are true invariants.

In II b (ii) we assumed Y_1 a principal solution and wrote Y_2 in the form $Y_2 = xY_1 + e^{\beta x}P_2(x)$. Noting that for the proper periodic B transformation $\phi(\bar{x}) = \bar{x} + P(\bar{x})$, P periodic (see end of § 1), we see that such a transformation will not change the form of Y_2. Clearly a proper periodic A transformation will not do so either. And neither of these will change the sign of $W(Y_1, Y_2)$, provided that for the B transformation $\phi' > 0$. Hence here also there is no possibility of further reduction.

To sum up: Under proper periodic transformations of types A and B where for type B $\phi' < 0$ there is no possibility of reducing the canonical forms found * to a smaller number and the quantities listed as invariants are actually such. Two equations of different classes or of the same class with different invariants cannot be transformed the one into the other: two equations of the same class with the same invariants may be transformed the one into the other.

8. *Further Restriction of the Transformations employed.* In the two cases treated below we used a transformation $x = \bar{x} + a$ to bring a zero of a principal solution to the origin or from the origin to some other point. If we restrict ourselves to those transformations which carry the interval $(0, 2\pi)$ into the same interval, it is still possible to reduce the equation to simple canonical forms, but we shall be obliged to distinguish further cases.

If the roots of the characteristic equation are equal and the principal solution vanishes at 0, we may use the transformation and the canonical forms suggested in the footnote on p. 605. If we employ only transformations carrying $(0, \pi)$ into $(0, 2\pi)$, it is clear that $Y_1(0) = 0$ and $Y_1(0) \neq 0$ are distinct cases, since our proper periodic transformations carry a principal solution into a principal solution.

If the roots of the characteristic equation are real and unequal and a principal solution has at least one zero, we must now distinguish four cases. Remembering that we assumed $\rho_2 \geq \rho_1$ and consequently $\alpha \geq 0$, thus definitely fixing ρ_2 and its associated principal solution, Y_2, we have:

(A) If $Y_2(0) = 0$, our work above holds and we have the canonical forms given.

(B) If $Y_1(0) = 0$, we may interchange Y_1 and Y_2 in (A). This is the same as changing the sign of α in our forms.

(C) Let a and b respectively be the smallest zeros of Y_2 and Y_1 in the interval $(0, 2\pi)$. Then if $0 < a < b$ we may in equation (21) on the right change $\bar{x}$ to $\bar{x} - \pi/2k$ and let the values $x = a$ and $\bar{x} = \pi/2k$ correspond. The transformation thus defined carries the interval $(a, a + 2\pi)$ into $(\pi/2k, 2\pi + \pi/2k)$ and by suitably choosing Y_2 on the left,—we may of course replace Y_2 by any constant multiple of itself,—the values $x = 0$ and $\bar{x} = 0$ will correspond. We will thus obtain the canonical forms of (A) with $\bar{x}$ replaced by $\bar{x} - \pi/2k$.

(D) Finally if $0 < b < a$, we may interchange Y_1 and Y_2 in (C) thus changing the sign of α.

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* We may of course choose other forms.

9. *Summary.* We conclude this paper by giving a summary of the different classes of equations considered together with the invariants * and the canonical forms for the solutions and equations of each class. Any linear differential equation of the second order whose coefficients are continuous in $(0, 2\pi)$ and periodic with period 2π may be reduced to one of these forms by transformations of the types

$$\text{(A)} \quad y = \lambda(x)\bar{y}, \qquad \text{(B)} \quad x = \phi(\bar{x}),$$

satisfying the conditions for being "proper" and periodic (§ 1) and carrying $(0, 2\pi)$ into $(0, 2\pi)$. A necessary and sufficient condition that two equations be reducible the one to the other by transformations of the types given above is that they belong to the same class and possess the same invariants.

I. Roots of characteristic equation real and distinct, ρ_1, ρ_2. Assume $\rho_2 > \rho_1$. Let $2\alpha = (1/2\pi) \log (\rho_2/\rho_1)$.

(i) Neither principal solution vanishes $0 \leq x < 2\pi$.

Invariant: ρ_2/ρ_1.

Canonical form of solutions: $e^{+\alpha x}$, $e^{-\alpha x}$,

Canonical form of equation: $y'' - \alpha^2 y = 0$.

(ii) A principal solution vanishes k times in the interval $0 \leq x < 2\pi$.

Invariants: ρ_2/ρ_1. The integer k.

The vanishing or not vanishing of a principal solution at 0, classified as follows:

(A) $Y_2(0) = 0$:

Canonical form of solutions:

$$\exp(+\alpha x) \sin (k/2)\bar{x} \cdot \exp[(2\alpha^2/k^2) \sin^2 (k/2)\bar{x}]$$
$$\exp(-\alpha\bar{x}) \cos (k/2)x \cdot \exp[(2\alpha^2/k^2) \cos^2 (k/2)\bar{x}].$$

Canonical form of equation: Equation (24).

(B) $Y_1(0) = 0$.

Canonical forms same as for (A) with $-\alpha$ in place of α.

(C) Let a and b be the smallest zeros of Y_2 and Y_2 respectively in the interval $(0, 2\pi)$ and $0 < a < b$.

Canonical forms same as for (A) with x replaced by $x - \pi/2k$.

(D) $0 < b < a$.

Canonical forms same as for (C) with $-\alpha$ in place of α.

II. Roots of characteristic equation real and equal.

a. Every solution a principal solution. Any solution vanishes exactly k times, $k \neq 0$, in the interval $0 \leq x < 2\pi$.

* The quantities named are essentially the only invariants. Of course any combinations or functions of them would also be invariants.

Invariant: k.

Canonical form of solutions: $\sin(k/2)x, \cos(k/2)x$.

Canonical form of equation: $y'' + (k^2/4)y = 0$.

b. There exists one and only one principal solution, Y_1.

(i) Y_1 does not vanish, $0 \leq x < 2\pi$.

Invariants: None.

Canonical form of solutions: $1, x$.

Canonical form of equation: $y'' = 0$.

(ii) Y_1 vanishes k times in the interval $0 \leq x < 2\pi$.

Invariants: k. Sign of $W(Y_1, Y_2)$, where Y_2 is any other solution written in the form $Y_2 = xY_1 + e^{\beta x}P(x)$, see § 5.

Vanishing or not vanishing of Y_1 at 0.

(A) $Y_1(0) \neq 0$.

Canonical form of solutions:

$$\cos(k/2)x, x\cos(k/2)x + \mu\sin(k/2)x,$$

where $\mu = 1$, if $W(Y_1, Y_2) > 0$;

$$\mu < 0 \text{ and } -\mu > 2/k, \text{ (e. g. } \mu = -3)$$

if $W(Y_1, Y_2) < 0$.

Canonical form of equation:

$$4y''[1 + \mu k + \cos kx] + 4ky' \sin kx + [3k^2 + \mu k^3 - k^2 \cos kx]y = 0.$$

(B) $Y_1(0) = 0$. Canonical form of solutions:

$$\sin(k/2)x, \; x\sin(k/2)x - \mu\cos(k/2)x.$$

Canonical form of equation:

$$4y''[1 + \mu k - \cos kx] - 4ky' \sin kx + y[3k^2 + \mu k^3 + k^2 \cos kx] = 0.$$

III. Roots of characteristic equation complex, $\rho(\cos\phi \pm i\sin\phi)$.

The solution vanishing at 0 vanishes n times in the interval $0 \leq x < 2\pi$.

If n is even, let $2k + 2 = n$ and take ϕ so that $\pi < \phi < 2\pi$.

If n is odd, let $2k + 1 = n$ and take ϕ so that $0 < \phi < \pi$.

Invariants: ϕ, determined as above. n (or k determined from it).

Canonical form of solutions:

$$\cos[(\phi/2\pi) + k]x, \quad \sin[(\phi/2\pi) + k]x.$$

Canonical form of equation:

$$y'' + [(\phi/2\pi) + k]^2 y = 0.$$

UNIVERSITY OF VERMONT,
BURLINGTON, VERMONT.

Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points.

By H. E. Buchanan.

Introduction. Since the announcement of the straight line and equilateral triangle solutions of the three body problem by Lagrange, many periodic orbits near these positions have been found. Most of these have been limited to the case in which one of the bodies is infinitesimal. The equilateral triangle configuration has received much less attention than the straight line cases probably because of the greater difficulty of the Algebra encountered. A special case of the problem of this paper was discussed by Professor Thomas Buck in his dissertation,* one of the bodies being infinitesimal. Professor Daniel Buchanan † has discussed the asymptotic orbits for an infinitesimal body near the equilateral triangle positions. The paper of Professor Buck has been valuable to the author as a check, and his results appear here as a special case.

The Equation of Variation. Let the units of mass and distance be chosen so that the gravitational constant and the distances between the masses are equal to unity. When the axes are rotating uniformly in the ξ, η plane, the positions in the triangle solutions are given by equations 57, page 310, Moulton's *Celestial Mechanics.* The square of the angular velocity is $\omega^2 = m_1 + m_2 + m_3$. We make the transformation—

$$(1) \qquad \xi_i = \xi_i^{(0)} + x_i', \quad \eta_i = \eta_i^{(0)} + y_i', \quad \zeta_i = z_i', \quad (i = 1, 2, 3).$$

Then the differential equations which define the variations become

$$\frac{d^2x_i'}{dt^2} - 2\omega \frac{dy_i'}{dt} = \omega^2(\xi_i^{(0)} + x_i') - \sum_1^3 m_j \frac{(x_i' + \xi_i^{(0)} - x_j' - \xi_j^{(0)})}{r^3_{ij}},$$

$$\frac{d^2y_i'}{dt^2} + 2\omega \frac{dx_i'}{dt} = \omega^2(\eta_i^{(0)} + y_i') - \sum_1^3 m_j \frac{(y_i' + \eta_i^{(0)} - y_j' - \eta_j^{(0)})}{r^3_{ij}},$$

$$\frac{d^2z_i'}{dt^2} = -\sum_1^3 m_j \frac{(z_i' - z_j')}{r^3_{ij}}, \qquad (i = 1, 2, 3, j \neq i).$$

* *Periodic Orbits*, Publication No. 161, Carnegie Institution.

† *Transactions of the Cambridge Philosophical Society*, Vol. 22 (1919).

Expansion of the Right Members. From this point on we omit the superscripts on $\xi_i^{(0)}$ and $\eta_i^{(0)}$ and wherever these symbols occur it will be understood that they refer to the equilateral triangle positions.

We expand the right members as power series in x_i', y_i' and z_i'. The x-axis is taken parallel to the line joining m_1 and m_2. Then $\xi_i - \xi_j$ can never vanish. Hence the regions of convergence will be determined by *

$$\frac{(y_i' + \eta_i - y_j' - \eta_j)^2 + (z_i' - z_j')^2}{(x_i' + \xi_i - x_j' - \xi_j)^2} < 1,$$

$$\left| \frac{x_i' - x_j'}{\xi_i - \xi_j} \right| < 1, \qquad (i = 1, 2, 3,\ j \neq i.)$$

Inside this region we may expand the right members as power series in x_i', y_i' and z_i' in the form

$$X_{1i} + X_{2i} + \cdots,$$
$$Y_{1i} + Y_{2i} + \cdots,$$
$$Z_{1i} + Z_{2i} + \cdots,$$

where each $X_{ji}\, Y_{ji}\, Z_{ji}$ is a homogeneous function of degree j in $x_i'\, y_i'\, z_i'$. Further X_{ji} and Y_{ji} are functions of $z_i'^2$ and Z_{ji} is z_i' times a function of $z_i'^2$.

The Parameters ϵ and δ. Applying the transformation

$$x_1' = \epsilon' x_i,\ y_i' = \epsilon' y_i,\ z_i' = \epsilon' z_i,\ t = (1+\delta)\tau, \text{ we find}$$

$$(d^2x_i/d\tau^2) - 2\omega(1+\delta)(dy_i/d\tau) = (1+\delta)^2\{X_{1i} + \epsilon' X_{2i} + \cdots\}$$

$$(2) \qquad (d^2y_i/d\tau^2) + 2\omega(1+\delta)(dx_i/d\tau) = (1+\delta)^2\{Y_{1i} + \epsilon' Y_{2i} + \cdots\}$$

$$(d^2z_i/d\tau^2) = (1+\delta)^2\{Z_{1i} + \epsilon' Z_{2i} + \cdots\},$$

$$(i = 1, 2, 3).$$

These equations are valid for the physical problem as long as the convergence of the right members is assured. We generalize the problem by replacing ϵ' by ϵ where ϵ may take all values in the vicinity of zero. We shall have a solution of the physical problem only when $\epsilon = \epsilon'$. The Method will be to find all periodic solutions of these when $\epsilon = \delta = 0$ and discuss the analytic continuation of these when ϵ increases to ϵ'.

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* Cf. H. E. Buchanan, "Periodic Orbits," *American Journal of Mathematics*, Vol. 45, No. 2 (1923), where a similar problem is treated.

The Center of Gravity Integrals. Since the origin is at the center of mass, we have

$$m_1x_1 + m_2x_2 + m_3x_3 = 0,$$
$$m_1y_1 + m_2y_2 + m_3y_3 = 0,$$
$$m_1z_1 + m_2z_2 + m_3z_3 = 0.$$

By means of these we eliminate the x_2, y_2, z_2 equations so that hereafter i has only the values 1 and 3.

It is evident that

$$\partial x_2/\partial x_1 = \partial y_2/\partial y_1 = \partial z_2/\partial z_1 = -m_1/m_2,$$
$$\partial x_2/\partial x_3 = \partial y_2/\partial y_3 = \partial z_2/\partial z_3 = -m_3/m_2.$$

The Linear Terms of the Differential Equations. We choose our axes so that the x-axis is parallel to the line joining m_1 and m_2 and m_1 is in the negative direction from m_2. It follows that

$$\xi_2 - \xi_1 = 1,\ \xi_3 - \xi_1 = \tfrac{1}{2},\ \xi_3 - \xi_2 = -\tfrac{1}{2},$$
$$\eta_2 - \eta_1 = 0,\ \eta_3 - \eta_1 = \eta_3 - \eta_2 = 3^{\frac{1}{2}}/2.$$

In equation (2) put $\epsilon = \delta = 0$. There results

$$(3)\quad \begin{cases} (d^2x_i/d\tau^2) - 2\omega(dy_i/d\tau) = A_{1i}x_1 + A_{3i}x_3 + B_{1i}y_1 + B_{3i}y_3, \\ (d^2y_i/d\tau^2) + 2\omega(dx_i/d\tau) = C_{1i}x_1 + C_{3i}x_3 + D_{1i}y_1 + D_{3i}y_3, \end{cases}$$

$$(4)\qquad d^2z_i/d\tau^2 = E_{1i}z_1 + E_{3i}z_3, \qquad (i = 1, 3),$$

where

$$A_{11} = 3\omega^2 - (9m_3/4),\ A_{31} = 9m_3/4,\ A_{13} = D_{13} = 0,$$
$$A_{33} = 3\omega^2/4,\ B_{11} = C_{11} = (3\cdot 3^{\frac{1}{2}}/4)m_3,\ B_{31} = C_{31} = -(3\cdot 3^{\frac{1}{2}}/4)m_3,$$
$$B_{13} = C_{13} = -(3\cdot 3^{\frac{1}{2}}/2)m_1,\ B_{33} = C_{33} = (3\cdot 3^{\frac{1}{2}}/2)m_1 - (3\cdot 3^{\frac{1}{2}}/4)\omega^2,$$
$$D_{11} = 9m_3/4,\ D_{31} = -(9m_3/4),\ D_{33} = 9\omega^2/4),\ E_{11} = E_{33} = -\omega^2,$$
$$E_{31} = E_{13} = 0.$$

Equations (3) and (4) are mutually independent. The solutions of (4) are

$$z_1 = M_{11}\cos\omega\tau + M_{12}\sin\omega\tau,$$
$$z_3 = M_{31}\cos\omega\tau + M_{33}\sin\omega\tau.$$

We consider the solutions of the xy-equations in subsequent paragraphs.

The Characteristic Equation. Let us substitute in (3)

$$x_i = K_i e^{\lambda\tau}, \qquad y_i = L_i e^{\lambda\tau}$$

there results a set of equations linear and homogeneous in K_i and L_i. In order that there be any solution different from $K_i = L_i = 0$ we must have

$$\begin{vmatrix} \lambda^2 - 3\omega^2 + \frac{9m_3}{4}, & -\frac{9m_3}{4}, & -2\lambda\omega - \frac{3\sqrt{3}\,m_3}{4}, & \frac{3\sqrt{3}\,m_3}{4} \\ 0, & \lambda^2 - \frac{3\omega^2}{4}, & \frac{3\sqrt{3}\,m_1}{2}, & -2\lambda\omega - \frac{3\sqrt{3}\,m_1}{2} + \frac{3\sqrt{3}\,\omega^2}{4} \\ 2\lambda\omega - \frac{3\sqrt{3}\,m_3}{4}, & \frac{3\sqrt{3}\,m_3}{4}, & \lambda^2 - \frac{9m_3}{4}, & \frac{9m_3}{4} \\ \frac{3\sqrt{3}m_1}{2}, & 2\lambda\omega - \frac{3\sqrt{3}\,m_1}{2} + \frac{3\sqrt{3}\,\omega^2}{4}, & 0, & \lambda^2 - \frac{9\omega^2}{4} \end{vmatrix} = 0. \tag{5}$$

The Roots $+\omega$ and $-\omega$.—Lagrange * has proved that there are elliptic orbits in which the bodies all remain at the vertices of an equilateral triangle. These ought to appear here as oscillations near the circular triangle solutions. Therefore we expect equation (5) to have $\lambda = \pm\omega$ as roots. Substituting $\lambda = +\omega$ we find

$$\begin{vmatrix} -4\omega^2 + 9m_3/4, & -9m_3/4, & -2\omega^2 - (3\cdot 3^{1/2}/4)m_3, & (3\cdot 3^{1/2}/4)m_3 \\ 0, & -7\omega^2/4, & (3\cdot 3^{1/2}/2)m_1, & -2\omega^2 - (3\cdot 3^{1/2}/2)m_1 + (3\cdot 3^{1/2}/4)\omega^2 \\ 2\omega^2 - (3\cdot 3^{1/2}/4)m_3, & (3\cdot 3^{1/2}/4)m_3, & -\omega^2 - 9m_3/4, & 9m_3/4 \\ (3\cdot 3^{1/2}/2)m_1, & 2\omega^2 - (3\cdot 3^{1/2}/2)m_1 + (3\cdot 3^{1/2}/4)\omega^2, & 0, & (-13\omega^2/4). \end{vmatrix}$$

Adding the second column plus 2ι times the sum of the third and fourth to the first column and then adding -2ι times the second row to the first row and expanding we find that this determinant vanishes. Hence $\lambda = \omega$ is a root. A similar computation shows that $\lambda = -\omega$ is also a root.

The Roots 0, 0. In a paper on Periodic Orbits near the Straight line solutions the author † found $\lambda = 0$ to be a double root of the characteristic equation. The physical situation being similar here it is natural to expect

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* Tisserand, *Méchanique Céleste*, Vol. 1, Chap. 8; Moulton, *Celestial Mechanics*, p. 217.

† *American Journal of Mathematics*, April, 1923.

that $\lambda = 0$ will be a double root. Direct substitution of 0 for λ gives, after considerable computation,

$$(27\omega^4/16)\begin{vmatrix} -4, & -9m_3/4, & 0, & (3\cdot 3^{1/2}/4)m_3 \\ 0, & -3m_1/2, & 0, & (-3\cdot 3^{1/2}/2)m_1 \\ 0, & (3\cdot 3^{1/2}/4)m_3, & 0, & 9m_3/4 \\ 1-3^{1/2}, & (-3/2)m_1+3\omega^2/4, & -1, & (-3\cdot 3^{1/2}\omega^2/4). \end{vmatrix}$$

This is obviously equal to

$$(-27/4)\omega^4 \begin{vmatrix} -3m_1/2, & (-3\cdot 3^{1/2}/2)m_1 \\ (3\cdot 3^{1/2}/4)m_3, & 9m_3/4, \end{vmatrix} = 0.$$

Hence $\lambda = 0$ is a root.

To show that $\lambda = 0$ is a double root we differentiate (5) with respect to λ and then set $\lambda = 0$. Discarding the common factor 2ω we have the sum of four third order determinants. One of these vanishes identically and the sum of the others reduces to

$$(81\cdot 3^{1/2}/8)m_1m_3\omega^2 + (81\cdot 3^{1/2}/16)m_3\omega^2(m_3-\omega^2) - (81\cdot 3^{1/2}/16)m_3\omega^2(-\omega^2+m_3+2m_1)$$

which vanishes. Hence $\lambda = 0$ is a double root.

The Conditions on the Masses. From the above discussion it follows that $\lambda^4+\omega^2\lambda^2$ must be a factor of equation (5). Expanding and dividing by $\lambda^4+\omega^2\lambda^2$ there is found

$$\lambda^4+\omega^2\lambda^2+(27/4)(m_1m_2+m_1m_3+m_2m_3)=0$$

whence

$$(6) \qquad \lambda^2 = \{-\omega^2 \pm [\omega^4-27(m_1m_2+m_1m_3+m_2m_3)]^{1/2}\}/2.$$

There are three cases which arise. They are expressed by the following conditions on the masses:

$$\text{I.} \quad \omega^4-27(m_1m_2+m_1m_3+m_2m_3)>0,$$
$$\text{II.} \quad \omega^4-27(m_1m_2+m_1m_3+m_2m_3)=0,$$
$$\text{III.} \quad \omega^4-27(m_1m_2+m_1m_3+m_2m_3)<0.$$

In case I the characteristic exponents are

$$\iota\omega, -\iota\omega, 0, 0, \iota\nu, -\iota\nu, \iota\rho, -\iota\rho$$

where $\nu = \{\omega^2 - [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}^{1/2}/2^{1/2}$,

and $\rho = \{\omega^2 + [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{1/2}\}^{1/2}/2^{1/2}$.

It follows that

$$\rho > \nu.$$

In case II the characteristic exponents are

$$\iota\omega, -\iota\omega, 0, 0, \iota\omega/2^{1/2}, \iota\omega/2^{1/2}, -\iota\omega/2^{1/2}, -\iota\omega/2^{1/2}.$$

In case III the exponents are

$$\iota\omega, -\iota\omega, 0, 0, \alpha - \iota\beta, -\alpha + \iota\beta, \alpha + \iota\beta, -\alpha - \iota\beta.$$

The Solutions of the xy Equations, Case I. With the characteristic exponents given above, the solutions of equations (3) are

$$(7)\quad \begin{cases} x_1 = K_{11}e^{\iota\omega\tau} + K_{12}e^{-\iota\omega\tau} + K_{13} + K_{14}\tau + K_{15}e^{\iota\nu\tau} + K_{16}e^{-\iota\nu\tau} + K_{17}e^{\iota\rho\tau} + K_{18}e^{-\iota\rho\tau} \\ x_3 = K_{31}e^{\iota\omega\tau} + K_{32}e^{-\iota\omega\tau} + K_{33} + K_{34}\tau + K_{35}e^{\iota\nu\tau} + K_{36}e^{-\iota\nu\tau} + K_{37}e^{\iota\rho\tau} + K_{38}e^{-\iota\rho\tau} \\ y_1 = L_{11}e^{\iota\omega\tau} + L_{12}e^{-\iota\omega\tau} + L_{13} + L_{14}\tau + L_{15}e^{\iota\nu\tau} + L_{16}e^{-\iota\nu\tau} + L_{17}e^{\iota\rho\tau} + L_{18}e^{-\iota\rho\tau} \\ y_3 = L_{31}e^{\iota\omega\tau} + L_{32}e^{-\iota\omega\tau} + L_{33} + L_{34}\tau + L_{35}e^{\iota\nu\tau} + L_{36}e^{-\iota\nu\tau} + L_{37}e^{\iota\rho\tau} + L_{38}e^{-\iota\rho\tau}. \end{cases}$$

There must be only eight arbitrary constants. Hence we choose K_{ij}, $j = 1 \cdots 8$ arbitrarily. Then K_{3j} and L_{ij}, $j = 1, 2, 5, 6, 7, 8$ are determined in terms of K_{1j} by any three of the equations,

$$(8)\quad \begin{cases} [\lambda^2 - \omega^2 + (9m_3/4)]K_{1j} - (9m_3/4)K_{3j} - [2\omega\lambda + (3\cdot 3^{1/2}/4)m_3]\,L_{1j} \\ \qquad + (3\cdot 3^{1/2}/4)m_3L_{3j} = 0, \\ [\lambda^2 - (3\omega^2/4)]K_{3j} + (3\cdot 3^{1/2}/2)m_1L_{1j} - [2\omega\lambda - (3\cdot 3^{1/2}/2)m_1 \\ \qquad + (3\cdot 3^{1/2}/4)\omega^2]L_{3j} = 0, \\ [2\lambda\omega - (3\cdot 3^{1/2}/4)m_3]K_{1j} + (3\cdot 3^{1/2}/4)m_3K_{3j} + [\lambda^2 - (9m_3/4)]\,L_{1j} \\ \qquad + (9m_3/4)L_{3j} = 0, \\ (3\cdot 3^{1/2}/2)m_1K_{1j} + [2\lambda\omega - (3\cdot 3^{1/2}/2)m_1 + (3\cdot 3^{1/2}/4)\omega^2]\,K_{3j} \\ \qquad + [\lambda^2 - (9\omega^2/4)]L_{3j} = 0. \end{cases}$$

To determine K_{33}, K_{34}, L_{13}, L_{14}, L_{33} and L_{34} we substitute in equations (3)

$$x_1 = K_{13} + K_{14}\tau, \qquad y_1 = L_{13} + L_{14}\tau,$$
$$x_3 = K_{33} + K_{34}\tau, \qquad y_3 = L_{33} + L_{34}\tau.$$

There results, since the equations must be identities in τ,

$$(9)\qquad \begin{cases} A_{11}K_{14} + A_{31}K_{34} + B_{11}L_{14} + B_{31}L_{34} = 0, \\ \qquad A_{33}K_{34} + B_{13}L_{14} + B_{33}L_{34} = 0, \\ B_{11}K_{14} + B_{31}K_{34} + D_{11}L_{14} + D_{31}L_{34} = 0, \\ B_{13}K_{14} + B_{33}K_{34} \qquad + D_{33}L_{34} = 0, \end{cases}$$

$$(9')\qquad \begin{cases} B_{31}K_{33} + A_{31}L_{13} - A_{31}L_{33} = 2\omega K_{14} - B_{11}K_{13}, \\ A_{31}K_{33} + B_{11}L_{13} + B_{31}L_{33} = -2\omega L_{14} - A_{11}K_{13}, \\ A_{33}K_{33} + B_{13}L_{13} + B_{33}L_{33} = -2\omega L_{34}, \\ B_{33}K_{33} \qquad + 3A_{33}L_{33} = 2\omega K_{34} - B_{13}K_{13}. \end{cases}$$

Solving the first, second and fourth equations of (9) for K_{34}, L_{14}, and L_{34} in terms of K_{14} there results

$$(10)\qquad K_{34} = -\frac{m_1 + m_2}{m_3} K_{14}, \qquad L_{14} = -\frac{2m_2 + m_3}{3^{\frac{1}{2}} m_3} K_{14},$$

$$L_{34} = \frac{m_1 - m_2}{3^{\frac{1}{2}} m_3} K_{14}.$$

These values satisfy the third equation identically. Substituting these in equations (9′) and solving the last three equations for K_{33}, L_{13} and L_{33} we have

$$(11)\qquad \begin{cases} K_{33} = -\dfrac{m_1 + m_2}{m_3} K_{13} + \dfrac{4m_2\omega}{3(3)^{\frac{1}{2}} {m_3}^2} K_{14}, \\ L_{13} = -\dfrac{2m_2 + m_3}{(3)^{\frac{1}{2}} m_3} K_{13} + \dfrac{8}{9} \dfrac{({m_3}^2 + {m_2}^2 + m_2 m_3}{{m_3}^2\omega} K_{14}, \\ L_{33} = \dfrac{m_1 - m_2}{(3)^{\frac{1}{2}} m_3} K_{13} - \dfrac{4}{9} \dfrac{(2m_3 m_1 + m_2\omega^2 - 2{m_2}^2)}{{m_3}^2\omega} K_{14}. \end{cases}$$

These values satisfy the first equation of (9′) identically.

The last three equations of (8) yield the following for K_{3j}, L_{1j} and L_{3j} in terms of K_{1j} $j = 1, 2, 5, 6, 7, 8$.

$$(12)\qquad \begin{aligned} K_{3j} &= (27m_1/8D)\,[(m_2 - m_1)\lambda^2 - 2\cdot 3^{\frac{1}{2}}\omega(m_1 + m_2)\lambda]K_{1j}, \\ L_{3j} &= (3\cdot 3^{\frac{1}{2}}m_1/2D)\{-\lambda^4 + \lambda^2\,[-(13\omega^2/4) + (9m_3/4)] \\ &\qquad + (3\cdot 3^{\frac{1}{2}}/2)\omega(m_1 - m_2)\lambda\}K_{1j}, \\ L_{1j} &= (K_{1j}/D)\,[-2\omega\lambda^3(\lambda^2 + \omega^2) + (3\cdot 3^{\frac{1}{2}}/8)m_3\lambda^2 \\ &\qquad (2\lambda^2 + 2\omega^2 + 9m_1) + (27\cdot 3^{\frac{1}{2}}/4)m_1 m_3\omega\lambda], \end{aligned}$$

where

$$D = \lambda^6 + \lambda^4[\omega^2 - (9m_3/4)] + (27/8)\lambda^2[2m_1m_2 + m_1m_3 - (2/3)m_3\omega^2] + (27 \cdot 3^{1/2}/4)\omega m_1 m_3 \lambda.$$

The Generating Solutions. The generating solutions must be periodic. Hence we have the following sub-cases of case I to consider.

Case I. (a) ρ, ω and ν incommensurable,

$$(1)\quad x_i = K_{i5}e^{i\nu\tau} + K_{i6}e^{-i\nu\tau}, \quad y_i = L_{i5}e^{i\nu\tau} + L_{i6}e^{-i\nu\tau}, \quad z_1 = 0,$$

$$(2)\quad x_i = K_{i7}e^{i\rho\tau} + K_{i8}e^{-i\rho\tau}, \quad y_i = L_{i7}e^{i\rho\tau} + L_{i8}e^{-i\rho\tau}, \quad z_i = 0,$$

$$(3)\quad x_i = K_{i1}e^{i\omega\tau} + K_{i2}e^{-i\omega\tau}, \quad y_i = L_{i1}e^{i\omega\tau} + L_{i2}e^{-i\omega\tau},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(b) ρ and ν commensurable with each other but not with ω.

$$(4)\quad \left.\begin{aligned} x_i &= K_{i5}e^{i\nu\tau} + K_{i6}e^{-i\nu\tau} + K_{i7}e^{i\rho\tau} + K_{i8}e^{-i\rho\tau}, \\ y_i &= L_{i5}e^{i\nu\tau} + L_{i6}e^{-i\nu\tau} + L_{i7}e^{i\rho\tau} + L_{i8}e^{-i\rho\tau}, \end{aligned}\right\}, \quad z_i = 0.$$

(c) ρ and ω commensurable but ν incommensurable with them.

$$(5)\quad \left.\begin{aligned} x_i &= K_{i1}e^{i\omega\tau} + K_{i2}e^{-i\omega\tau} + K_{i7}e^{i\rho\tau} + K_{i8}e^{-i\rho\tau}, \\ y_i &= L_{i1}e^{i\omega\tau} + L_{i2}e^{-i\omega\tau} + L_{i7}e^{i\rho\tau} + L_{i8}e^{-i\rho\tau}, \end{aligned}\right\},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(d) ν and ω commensurable but ρ incommensurable with them.

$$(6)\quad \left.\begin{aligned} x_i &= K_{i1}e^{i\omega\tau} + K_{i2}e^{-i\omega\tau} + K_{i5}e^{i\nu\tau} + K_{i6}e^{-i\nu\tau} \\ y_i &= L_{i1}e^{i\omega\tau} + L_{i2}e^{-i\omega\tau} + L_{i5}e^{i\nu\tau} + L_{i6}e^{-i\nu\tau} \end{aligned}\right\},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

(e) ρ, ω and ν all commensurable.

$$(7)\quad x_i = K_{i1}e^{i\omega\tau} + K_{i2}e^{-i\omega\tau} + K_{i5}e^{i\nu\tau} + K_{i6}e^{-i\nu\tau} + K_{i7}e^{i\rho\tau} + K_{i8}e^{-i\rho\tau},$$
$$y_i = L_{i1}e^{i\omega\tau} + L_{i2}e^{-i\omega\tau} + L_{i5}e^{i\nu\tau} + L_{i6}e^{-i\nu\tau} + L_{i7}e^{i\rho\tau} + L_{i8}e^{-i\rho\tau},$$
$$z_i = M_{i1}\cos\omega\tau + M_{i2}\sin\omega\tau.$$

Transformation to the Normal Form. It will be convenient to make a transformation of variables of the form

$$x_i = \sum_{j=1}^{8} a_{1j}u_j, \quad dx_i/dt = \sum_{j=1}^{8} a_{2j}u_j, \quad x_3 = \sum_{j=1}^{8} a_{3j}u_j,$$

$$dx_3/dt = \sum_{j=1}^{8} a_{4j}u_j, \quad y_1 = \sum_{j=1}^{8} a_{5j}u_j, \quad dy_1/dt = \sum_{j=1}^{8} a_{6j}u_j,$$

$$y_3 = \sum_{j=1}^{8} a_{7j}u_j, \quad dy_3/dt = \sum_{j=1}^{8} a_{8j}u_j.$$

It is desired to determine the a_{ij} so that the differential equations (2) take the form

$$\begin{aligned}
u_1' &= (1+\delta)\omega\imath u_1 + (1+\delta)\epsilon P_1(u_j, z_i),\\
u_2' &= -(1+\delta)\omega\imath u_2 + (1+\delta)\epsilon P_2(u_j, z_i),\\
u_3' &= (1+\delta)u_4 + (1+\delta)\epsilon P_3(u_j, z_i),\\
u_4' &= \qquad\qquad (1+\delta)\epsilon P_4(u_j, z_i),\\
u_5' &= (1+\delta)\nu\imath u_5 + (1+\delta)\epsilon P_5(u_j, z_i),\\
u_6' &= -(1+\delta)\nu\imath u_6 + (1+\delta)\epsilon P_6(u_j, z_i),\\
u_7' &= (1+\delta)\rho\imath u_7 + (1+\delta)\epsilon P_7(u_j, z_i),\\
u_8' &= -(1+\delta)\rho\imath u_8 + (1+\delta)\epsilon P_8(u_j, z_i),
\end{aligned} \tag{13}$$

where the P_j are power series in u_j beginning with terms of the second degree. To determine the a_{ij} integrate equations (13) after all the terms of the right members have been dropped except the linear ones. The result is

$$u_1 = K_{11}e^{(1+\delta)\omega\imath\tau}, \quad u_2 = K_{12}e^{-(1+\delta)\omega\imath\tau}, \quad u_3 = K_{13} + (1+\delta)K_{14}\tau,$$

$$u_4 = K_{14}, \quad u_5 = K_{15}e^{(1+\delta)\nu\imath\tau}, \quad u_6 = K_{16}e^{-(1+\delta)\nu\imath\tau},$$

$$u_7 = K_{17}e^{(1+\delta)\rho\imath\tau}, \quad u_8 = K_{18}e^{-(1+\delta)\rho\imath\tau}.$$

Adding these eight equations we get the value of x_1 obtained from the linear terms of equation (2). The values of x_1', x_3, x_3', y_i and y_i' are obtained from x_1 by multiplying the K_{ij} by certain constants given in equations (12). We may choose all the K_{ij} equal to unity. This gives the following form for the transformation:

$$
(14)\quad \begin{cases}
x_1 = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8, \\
x_1' = \omega\iota(u_1 - u_2) + u_4 + \nu\iota(u_5 - u_6) + \rho\iota(u_7 - u_8), \\
x_3 = a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + a_{34}u_4 + a_{35}u_5 + a_{36}u_6 + a_{37}u_7 + a_{38}u_8, \\
x_3' = \omega\iota(a_{31}u_1 - a_{32}u_2) + a_{34}u_4 + \nu\iota(a_{35}u_5 - a_{36}u_6) + \rho\iota(a_{37}u_7 - a_{38}u_8), \\
y_1 = a_{51}u_1 + a_{52}u_2 + a_{53}u_3 + a_{54}u_4 + a_{55}u_5 + a_{56}u_6 + a_{57}u_7 + a_{58}u_8, \\
y_1' = \omega\iota(a_{51}u_1 - a_{52}u_2) + a_{54}u_4 + \nu\iota(a_{55}u_5 - a_{56}u_6) + \rho\iota(a_{57}u_7 - a_{58}u_8), \\
y_3 = a_{71}u_1 + a_{72}u_2 + a_{73}u_3 + a_{74}u_4 + a_{75}u_5 + a_{76}u_6 + a_{77}u_7 + a_{78}u_8, \\
y_3' = \omega\iota(a_{71}u_1 - a_{72}u_2) + a_{74}u_4 + \nu\iota(a_{75}u_5 - a_{76}u_6) + \rho\iota(a_{77}u_7 - a_{78}u_8).
\end{cases}
$$

Orbits with Period $2\pi/\nu$. We choose to investigate Case I, (a), (1). In the normal variables the generating solution is

$$u_5 = K_5 e^{\nu\iota\tau}, \quad u_6 = K_6 e^{-\iota\nu\tau}, \quad u_j = 0, \quad (j = 1, 2, 3, 4, 7, 8),$$

$$z_i = 0.$$

We shall prove that these orbits do not exist outside the xy-plane. Let the initial conditions be

$$u_5 = k_5 + a_5, \; u_6 = k_6 + a_6, \; u_j = a_j, \; j = 1, 2, 3, 4, 7, 8,$$

$$z_i = g_i, \; z'_i = h_i, \; i = 1, 3 \text{ at } \tau = 0.$$

It is evident that if $g_i = h_i = 0$, $i = 1, 3$, the orbits, if they exist at all, must lie wholly in the xy-plane.

The conditions that the solutions be periodic are

$$(15)\qquad u_j(2\pi/\nu) - u_j(0) = 0, \quad z_i(2\pi/\nu) - z_i(0) = 0,$$

$$z_i'(2\pi/\nu) - z_i'(0) = 0.$$

We integrate equations (13) as power series in the initial constants, ϵ and δ. The linear terms of the z_i equations are independent of the u_i equations. Integrating the linear terms of the z_i equations and imposing the initial conditions,

$$z_{10} = g_1 \cos\omega\tau + (h_1/\omega)\sin\omega\tau, \quad z'_{10} = -\omega g_1 \sin\omega\tau + h_1 \cos\omega\tau,$$

$$z_{30} = g_3 \cos\omega\tau + (h_3/\omega)\sin\omega\tau, \quad z'_{30} = -\omega g_3 \sin\omega\tau + h_3 \cos\omega\tau.$$

Then the periodicity conditions for the z_i equations are:

$$0 = g_1[\cos(2\pi\omega/\nu) - 1] + (h_1/\omega)\sin(2\pi\omega/\nu) + \epsilon Q_1(a_j, g_1, h_i, \epsilon, \delta),$$

$$0 = -g_1\omega\sin(2\pi\omega/\nu) + h_1[\cos(2\pi\omega/\nu) - 1] + \epsilon Q_1'(a_j, g_i, h_i, \epsilon, \delta),$$

$$0 = g_3[\cos(2\pi\omega/\nu) - 1] + (h_3/\omega)\sin(2\pi\omega/\nu) + \epsilon Q_3(a_j, g_i, h_i, \epsilon, \delta),$$

$$0 = -g_3\sin(2\pi\omega/\nu) + h_3[\cos(2\pi\omega/\nu) - 1] + \epsilon Q_3'(a_j, g_i, h_i, \epsilon, \delta).$$

In the first three of these equations the determinant of the linear terms in g_1, h_1, h_3 does not vanish. There is a term independent of g_1, h_1, h_3. Hence * we may solve uniquely for g_1, h_1, and h_3 in terms of a_j, g_3, ϵ, δ. Substitute these solutions in the last equation. Since all the terms of the right members of the z_i equations of equation (2) carry either z_1 or z_3 as a factor, it follows that after the values of g_1, h_1, h_3 have been substituted, the fourth equation will carry g_3 as a factor. Removing this factor leaves a term independent of a_i, ϵ and δ. Therefore solution for g_3 in terms of a_i, ϵ, δ vanishing with these parameters is impossible and the orbits do not exist outside the xy-plane.

Suppression of the u_4 and u_6 Equations. The periodicity equations now become

$$u_j[(2\pi/\nu)] - u_j(0) = 0, \qquad (j = 1, \cdots, 8).$$

These are not independent. They are related by the integral of areas and the energy integral.

$$F_1 = \sum_{i=1}^{3} m_i(\bar{x}_i\bar{y}_i' - \bar{y}_i\bar{x}_i') - c_1 = 0,$$

$$F_4 = \tfrac{1}{2}\sum_{i=1}^{3}[(\bar{x}_i')^2 + (\bar{y}_i')^2 + (\bar{z}_i')^2] - h = 0.$$

These variables are related to the u_i equations by the following transformations:

$$\bar{x}_i = \xi_i\cos\omega t - \eta_i\sin\omega t,$$

$$\bar{y}_i = \xi_i\sin\omega t + \eta_i\cos\omega t,$$

$$\xi_i = \xi_i^{(0)} + \epsilon x_i,\ \eta_i = \eta_i^{(0)} + \epsilon y_i,\ \bar{z}_i = \epsilon z_i,$$

$$t = (1+\delta)\tau, \text{ and equations (14).}$$

Now let

$$u_j = a_j + s_j,\ (j = 1, 2, 3, 4, 7, 8), \quad u_5 = (k_5 + a_5)e^{(1+\delta)\nu i\tau} + s_5,$$

$$u_6 = (k_6 + a_6)e^{-(1+\delta)\nu i\tau} + s_6.$$

---

* Moulton, *Periodic Orbits*, Chap. I, page 1.

It follows that the $s_j = 0$ for $\tau = 0$. Since c_1 and h can be expressed in terms of the initial constants, we may consider F_1 and F_4 as functions of s_j and a_j vanishing with s_j whatever values a_j may have. Therefore they may be solved for s_4 and s_6 as power series in s_j, $(j = 1, 2, 3, 5, 7, 8)$, provided the Jacobian

$$J = \begin{vmatrix} \partial F_1/\partial s_4, & \partial F_1/\partial s_6 \\ \partial F_4/\partial s_4, & \partial F_4/\partial s_6 \end{vmatrix} \neq 0,$$

for zero values of the arguments.

The elements of J are found by direct computation to be

$$\partial F_1/\partial s_6 = -2m_1\omega - m_3\omega K_{35} + m_3\omega\, 3^{\frac{1}{2}}\, L_{35},$$
$$\partial F_1/\partial s_4 = -2m_1\omega - m_3\omega K_{34} + m_3\omega\, 3^{\frac{1}{2}}\, L_{34},$$
$$\partial F_4/\partial s_4 = 2\omega^2\{\xi_1 - (m_1/m_2)\xi_2 + [\xi_3 - (m_3/m_2)\xi_2]K_{34} + [\eta_1 - (m_1/m_2)\eta_2]L_{14} + [\eta_3 - (m_3/m_2)\eta_2]L_{34}\}.$$

The values of K_{34} and L_{34} are given in equations (10). When these are substituted $\partial F_1/\partial s_4$ vanishes, which makes it unnecessary to compute $\partial F_4/\partial s_6$. Supplying the values of K_{34}, L_{34} and L_{14} in $\partial F_4/\partial s_4$ we find

$$\frac{\partial F_4}{\partial s_4} = 2\omega^2\left[\frac{3m_3 - \omega^2}{m_3}\,\xi_1 - \frac{m_1 - m_2}{m_3}\,(3)^{\frac{1}{2}}\eta_1 + \tfrac{1}{2}\,\frac{m_1 + m_2}{m_2}\right].$$

Now if we choose $m_3 \leqq m_2 \leqq m_1$, then $3m_3 - \omega^2$ is negative or zero, $m_1 - m_2$ is positive or zero. Since ξ_1 and η_1 are negative it follows that one factor of J does not vanish.

Substituting the values of K_{35} and L_{35} from (12) in $\partial F_1/\partial s_6$ we find

$$\frac{\partial F_1}{\partial s_6} = -\frac{16\lambda^6 + 16\omega^2\lambda^4 + 81(m_1m_2 + m_1m_3 + m_2m_3)\lambda^2}{8D} \tag{16}$$

When λ^2 is taken from (6) this reduces to

$$\frac{\partial F_1}{\partial s_6} = \frac{27\{-\omega^2 + [\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3)]^{\frac{1}{2}}\}(m_1m_2 + m_1m_3 + m_2m_3)}{16D}$$

which cannot vanish. Therefore we may suppress the u_4 and u_6 equations.

Existence of Orbits with Period $2\pi/\nu$. With the initial conditions $u_j = a_j$ $(j = 1, 2, 3, 4, 7, 8)$, $U_5 = K_5 + a_5$, $U_6 = K_6 + a_6$, at $\tau = 0$, we may integrate equations (13) as power series in the initial constants, ϵ and δ in the form:

$$
\begin{aligned}
u_1 &= a_1 e^{(1+\delta)\omega\iota\tau} + \epsilon P_1(a_j, \epsilon, \delta),\\
u_2 &= a_2 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_2(a_j, \epsilon, \delta),\\
u_3 &= a_3 + a_4\tau + \epsilon P_3(a_j, \epsilon, \delta),\\
u_4 &= \quad a_4 + \epsilon P_4(a_j, \epsilon, \delta),\\
u_5 &= (K_5 + a_5) e^{(1+\delta)\nu\iota\tau} + \epsilon P_5(a_j, \epsilon, \delta),\\
u_6 &= (K_6 + a_6) e^{-(1+\delta)\nu\iota\tau} + \epsilon P_6(a_j, \epsilon, \delta),\\
u_7 &= a_7 e^{(1+\delta)\rho\iota\tau} + \epsilon P_7(a_j, \epsilon, \delta),\\
u_8 &= a_8 e^{-(1+\delta)\rho\iota\tau} + \epsilon P_8(a_j, \epsilon, \delta).
\end{aligned}
$$

The periodicity conditions are now

$$u_j(2\pi/\nu) - u_j(0), \quad (j = 1, 2, 3, 5, 7, 8),$$

since the u_4 and u_6 equations have been suppressed. There are six equations and ten parameters. Therefore we may choose a_4, a_5, a_6 and ϵ arbitrarily and solve for a_1, a_2, a_3, a_7, a_8 and δ in terms of the arbitrarily chosen ones. The solutions will vanish with these parameters. This solution is uniquely possible since the determinant of the linear terms is

$$K_5\{e^{\omega\iota(2\pi/\nu)} - 1\}\{e^{-\omega\iota(2\pi/\nu)} - 1\}\{e^{\rho\iota(2\pi/\nu)} - 1\}\{e^{-\rho\iota(2\pi/\nu)} - 1\}(2\pi/\nu) \neq 0$$

and since $a_1 = a_2 = a_3 = a_7 = a_8 = \delta = 0$ is not a solution. This proves the existence of an unique set of periodic orbits in the xy-plane with period $2\pi/\nu$. We could have used the period $2K\pi/\nu$ just as well and found an unique set of that period. Hence all these orbits are re-entrant after one revolution.

Orbits of Period $2\pi/\rho$. Following the method of the last two paragraphs we may establish the existence of plane orbits of period $-2\pi/\rho$. The steps are:

1. The initial conditions are $u_j = a_j$, $j = 1 \cdots 6$, $u_7 = k_7 + a_7$, $u_8 = k_8 + a_8$ at $\tau = 0$.

2. The orbits exist only in the xy-plane. The proof is exactly the same as the preceding case.

3. The u_4 and u_8 equations of the periodicity conditions may be suppressed. The only difference being in $\partial F_1/\partial s_8$ which may be found from $\partial F_1/\partial s_6$, equation (16), by replacing λ^2 by $-\rho^2$ from equation (6). There results

$$\partial F_1/\partial s_8 = 27M/D[\omega^2 + (\omega^4 - 27M)^{1/2}], \text{ where } M = m_1m_2 + m_2m_3 + m_3m_1.$$

Hence the u_4 and u_8 equations may be suppressed.

4. The remaining periodicity equations may be solved and an unique set of orbits of period $2\pi/\rho$ re-entrant after one revolution is established.

Conclusion. Thus we have established the existence of two types of periodic orbits, both of which lie in the plane of motion of the finite bodies. Since the linear terms of the z-equations give only the period $2\pi/\omega$ one expects that there will exist no periodic orbits in space except those of period $2\pi/\omega$. The orbits discussed exist if the condition on the masses

$$\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) > 0$$

is satisfied. The discussion of the construction of the solutions and the further geometrical properties of the orbits whose existence is proved as well as the other sub-cases is reserved for a later paper.

TULANE UNIVERSITY,
JUNE, 1927.

The Problem of Plane Involutions of Order $t > 2$.

By F. R. Sharpe.

1. *Introduction.* A plane involution of order t, I_t, is defined as the arranging of the points of a plane (x), by the methods of algebraic projective geometry, in groups of t points in such a way that a general point occurs in only one group. G. Castelnuovo * has proved that any I_t of this kind is rational. The groups can therefore be put into $(1, 1)$ correspondence with the points of a plane (z) by a transformation which can be expressed by equations of the form

$$z_i = f_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3,) \tag{1}$$

where the $f_i(x)$ are homogeneous polynomials in x_1, x_2, x_3 having no common factor and with a functional determinant which does not vanish identically. Transformations (1) which can be changed to the same form by a Cremona transformation of the (x) or (z) plane or of both planes, are considered to be equivalent, and when the functions $f_i(x)$ in (1) are reduced to a minimum order in this way the transformation is said to be reduced to an irreducible normal form. By (1), to the net of lines in (z)

$$a_1z_1 + a_2z_2 + a_3z_3 = 0 \tag{2}$$

corresponds the projective net of curves

$$a_1f_1 + a_2f_2 + a_3f_3 = 0. \tag{3}$$

Since 2 lines of (1) determine a point P on (z), the corresponding curves of (3) must meet in t variable points, the group of I_t corresponding to P. The transformation (1) is said to map the plane (x) on a t-fold plane (z). The problem of plane involutions of order t is to determine the irreducible normal forms of nets of curves (3) with t variable intersections. To define a group of points algebraically two equations are necessary. From the solutions by E. E. Kummer † of the analogous problem of determining a line of a $(1, n)$ congruence in $S_3(y)$, (1) being the analogue of the planes through a point, there are just two other ways of defining groups of points in the plane, (a)

* G. Castelnuovo, *Mathematische Annalen*, Vol. 44 (1894).

† E. E. Kummer, *Berlin Abhandlungen* (1866).

by the intersections of two projective nets of curves,* (b) by the intersection of the curves of a pencil with the algebraically corresponding curves of a linear system. The type (a) is analogous to the bisecants of a cubic curve and (b) to the lines meeting a fixed line l and a fixed curve C_{2n-2} meeting l in $2n-3$ points. If the linear system in (b) is of dimension $r > 2$ † then I_t belongs to a family of I_t, n being arbitrary. In the case of both (a) and (b) the corresponding I_t's can be reduced to the normal form (1) but are more easily discovered when in the form (a) or (b). It is now clear that the problem of plane involutions is closely connected with the theory of linear systems of plane curves. It will be shown that if the pencil of curves in (b) is of genus p then a finite number of irreducible normal forms of I_t exist for each $p \leq (t-1)(t-2)/2$. The actual determination of forms of I_t requires a knowledge of the linear systems of curves of least order of genus $p \leq t(t-1)/2$. A fairly complete determination is made for $t = 3$ and a less complete one for $t = 4$. The mapping of the plane (x) on a space $S_r(y)$, $r > 2$ is shown to give valuable information concerning possible forms of I_t.

2. *Properties of linear systems of curves* ‡ (G. Castelnuovo). Consider a system of curves of dimension r and order d

$$\sum_{i=1}^{r+1} a_i f_i(x) = 0 \tag{4}$$

determined by the basic points, a k-fold basis point being equivalent to $k(k+1)/2$ conditions so that

$$r - [d(d+3)]/2 + \sum k(k+1)/2 \geq 0 = s, \text{ the superabundance.} \tag{5}$$

The genus $\pi = (d-1)(d-2)/2 - \sum k(k-1)/2$

and the grade (number of variable intersections) $n = d^2 - \sum k^2$. Hence

$$r = n - \pi + 1 + s. \tag{6}$$

The virtual dimension ${}_v r = n - \pi + 1.$

For a regular system ${}_v r = r$ and $s = 0.$

It is known that $s \leq \pi - r + 1.$

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* A. M. Howe, T. R. Hollcroft, *American Journal of Mathematics*, Vols. 41, 44.

† If $r = 2$ the involution is reducible to a single type not containing a parameter n.

‡ G. Castelnuovo, *Torino Memorie*, Ser. 2, Vol. 42 (1891); G. Jung, *Annali di Matematico*, Ser. 2, Vol. 15 (1887), pp. 277-312; Ser. 2, Vol. 16 (1888), pp. 291-326.

Given two linear systems $|C_1|$, $|C_2|$, their sum $|C_1| + |C_2|$ is the system of order $d_1 + d_2$ with a $k_1 + k_2$ fold point at a point which is k_1, k_2 fold for $|C_1|$, $|C_2|$, respectively. If a curve of $|C_1|$ meets a curve of $|C_2|$ in t variable points, $[C_1, C_2] = t$, then $|C_1| + |C_2|$ is of grade $n_1 + n_2 + 2t$, of genus $\pi_1 + \pi_2 + t - 1$ and virtual dimension ${}_v r_1 + {}_v r_2 + t$.

3. *Properties of linear series of points on a fixed curve C of genus p and order m* * (L. Berzolari). The system of adjoint curves of order $m-3$ having a $k-1$ fold point at a k-fold point of C is of dimension $p-1$ and cuts on C groups of $2p-2$ variable points the canonical G^{p-1}_{2p-2}. A G_n^r included in the G^{p-1}_{2p-2} is special and $n \geq r(r+p+1)/(1+r)$ (A. Clebsch† with $r > n-p$. For a non-special G_n^r, $r = n-p$. The residual G_{2p-2-n} is of dimension $r+p-1-n$ by the Riemann-Roch theorem. The curves of a linear system of grade n and dimension r cut on a curve of the system a G_n^{r-1}. If $n > 2p-2$ then $r = n-p+1$, if $n \leq 2p-2$ the G_n^{r-1} can be cut by the adjoint curves of order $m-3$ and $r > n-p+1$ (C. Segre)‡ subject to $n \geq (r+1)(r+p)/r$ for curves of general moduli.

4. *Types of plane involutions I_t.* Type (b). For a given pencil of curves $|C|$ of genus p, $s = p$. Hence if we consider the t variable intersections of a curve of C with the curves of a second system $|C| + |\bar{C}|$ there are two cases, (1) C of grade p with $[C, \bar{C}] = t - p$, (2) $|C|$ of grade 0 with $[C, \bar{C}] = t$. If the system $|C| + |\bar{C}|$ represents

$$(a_1C_1 + a_2C_2)\bar{C} + a_3C_3 + \cdots a_{r+1}C_{r+1} = 0 \tag{7}$$

the involution I_t is determined by

$$\begin{gathered} z_2C_1 - z_1C_2 = 0 \\ (b_1z_3 + c_1)C_1\bar{C} + (b_2z_3 + c_2)C_2\bar{C} + (b_3z_3 + c_3)C_3 \\ + \cdots + (b_{r+1}z_3 + c_{r+1})C_{r+1} = 0 \end{gathered} \tag{8}$$

where b_i and c_i are of arbitrary orders $n-1$, n in z_1, z_2. Solving for $z_1 : z_2 : z_3$, we find for I_t in the normal form (1)

$$z_1 = C_1u, \quad z = C_2u, \quad z_3 = v, \tag{9}$$

where u and v are linear in $C_1\bar{C}$, $C_2\bar{C}$, $C_3, \cdots, C_{r+1}$ with coefficients of arbi-

* L. Berzolari, *Pascal Repertorium*, Vol. 2, p. 312 (1910).

† A. Clebsch, *Geometric*, Vol. 3, p. 37.

‡ C. Segre, *Rendiconti di Circolo Matematico Palermo*, Vol. I.

trary orders $n-1$, n respectively in C_1, C_2. The form (9) therefore determines for $r > 2$ a family of I_t.

Type (a). If two projective nets of curves belong to linear systems of grade n_1, n_2, genus p_1, p_2, virtual dimensions r_1, r_2, then $|C_1| + |C_2|$ is of grade $n_1 + n_2 + 2t$, genus $p_1 + p_2 + t - 1$ and virtual dimension $r_1 + r_2 + t$. The net defining the involution must therefore have $n_1 + n_2 + t$ additional simple basis points and has therefore superabundance

$$(n_1 + n + t) - 2 - (r_1 + r_2 + t) = p_1 + p_2.$$

The case of a net of lines and a net of general curves of order t always exists and the net defining the involution is of genus $t(t-1)/2$ and superabundance $(t-1)(t-2)/2$.

Type (1). In order that a given linear system of curves of grade N, genus p, dimension $r > 2$ determine an I_t, it must be possible to determine $N - t$ additional simple basis points so that through them passes a net of curves of the system. For a superabundant net, the curves of the net cut on a curve of the net a G'_t which must be included in the canonical G^{p-1}_{2p-2}. For curves of general moduli therefore $t \geq (p+2)/2$ that is

$$p \leq 2t - 2.$$

5. *The mapping of an I_t on $S_3(y)$.* For Type (b) we may take for $r > 2$

(10) $$y_1 = C_1\bar{C}, \quad y_2 = C_2\bar{C}, \quad y_3 = C_3, \quad y_4 = C_4 \text{ where } C_3 = 0,$$

$C_4 = 0$ are curves of the system $|C| + |\bar{C}|$. If $|C| + |\bar{C}|$ is of grade N then (10) represents a surface $F_N{}^3$, of order N, having $l \equiv y_1 = y_2 = 0$ for $N - t$-fold line, $\bar{C}$ being the image of l. The plane sections of $F_N{}^3$ through l are C_s the images of the curves of the pencil $a_1C_1 + a_2C_2 = 0$. Hence

(11) $$p \leq (t-1)(t-2)/2.$$

The surface $F_N{}^3$ may have other multiple curves or points necessary for its rationality. The adjoint surfaces of order $N - 3$ meet F in l and the other multiple curves or points and in variable curves. The image of the variable curves is the system of adjoint curves of order $d - 3$ of the system $|C| + |\bar{C}|$ and the residue of $(N-3)(|C| + |\bar{C}|)$ consists of the image curves of l and the other multiple curves and points of $F_N{}^3$. The involution I_t on $F_N{}^3$ is cut by the lines of the rational congruence defined by the image of the net

(12) $$c_1C_1u + c_2C_2u + c_3v = 0.$$

The involution may also be mapped in $S_r(y)$ on an $F_N{}^r$ by the equations

$$(13) \qquad y_1 = C_1\bar{C}, \quad y_2 = C_2\bar{C}, \quad y_3 = C_3, \cdots, y_{r+1} = C_{r+1}.$$

The pencil of $S_{r-1}(y)a_1y_2 + a_2y_2 = 0$ passes through a fixed C_{N-t} in the S_{r-2}, $y_1 = y_2 = 0$ and meets $F_N{}^r$ in residual curves C_t. Two V_{r-1}'s of the net corresponding to $c_1C_1u + c_2C_2u + c_3v = 0$ meet in a group of I_t on $F_N{}^r$. The representation on $F_N{}^3$ is a special case found by projecting from $S(_ry)$ into $S_3(y)$. For Type (1) we may take

$$(14) \qquad y_1 = C_1\bar{C}, \quad y_2 = C_2\bar{C}, \quad y_3 = C_3\bar{C}, \quad y_4 = C_4,$$

where $\bar{C}$ may be 1 of $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ belong to a linear system of dimension > 2 with possibly fewer simple basis points. In the case $\bar{C} = 1$, $F_N{}^3$ has a multiple $N - t$-fold curve. In the case $\bar{C} \neq 1$, $C_4 = 0$ belongs to a linear system including the net $a_1C_1 + a_2C_2 + a_3C_3 = 0$ with a residual $\bar{C}$. In this case $F_N{}^3$ has an $N - t$-fold point $(0, 0, 0, 1)$ whose image is a fundamental curve of $|C| + |\bar{C}|$.

Type (a) may be reduced to the form (1) and treated similarly. If the two nets belong to the same linear system of dimension 3 we may take $y_2 = C_2$, $(i = 1 \text{ to } 4)$. In this case $N = t$ and the involution I_t on $F_t{}^3$ is cut by the bisecants of a cubic curve.

6. *The known I_2* (M. Noether).* For subsequent use the known types of I_2 are derived by the methods developed in the preceding articles.

Type (b). $p = 0$, $a_1x_1 + a_2x_2 = 0$ †
$$(a_1x_1 + a_2x_2)x_3 + a_3x_3{}^2 + a_4x_1{}^2 + a_5x_1x_2 + a_6x_2{}^2 = 0.$$

The involution defined by

$$z_1 = x_1u, \quad z_2 = x_2u, \quad z_3 = v$$

is easily seen to be defined by the net of curves

$$C_n : A^{n-2}(4n - 6)B$$

the point A being $(0, 0, 1)$ and the $4n - 6$ simple basis points B given by $u = v = 0$.

Type (b). $p = 1$, $a_1C_1 + a_2C_2 = 0$ cubics $C_3 : 8AB$
$$(a_1C_1 + a_2C_2)C_1 + a_3C_6 : 8A^2 = 0.$$

* M. Noether, *Erlangen Medico Berichte* (1878).
† M. Noether, *Mathematische Annalen*, Vol. 3 (1871), p. 161.

The involution is defined by the net of curves $C_6 : 8A^2 2B$ the points B being given by $C_1 = 0$, $C_6 : 8A^2 = 0$.

Type (a). $a_1x_1 + a_2x_2 + a_3x_3 = 0.$
$a_1C_1 + a_2C_2 + a_3C_3 = 0$ (conics).

The I_2 is therefore defined by the net of cubics $C_3 : 7A$. There is a special case of the first Type (b) for $n = 2$

$$(a_1x_1 + a_2x_2)x_1 + a_3x_3^2 - a_3x_2^2 = 0.$$

The points A and the two points B are collinear. In the case of I_2 there is an involutorial transformation of the (x) plane which interchanges the 2 points of a group. These transformations in the order in which the 4 types are given, are known as the Jonquières, Bertini, Geiser, and the harmonic homology.*

7. *Families of I_3.* Type (b), $p = 0$, gives as in the preceding case

$$C_n : A^{n-3}(6n - 12)B.$$

Type (b). $p = 1$ with the pencil $C_3 : 8AB$.†

We proceed to determine the different linear systems (7) which include $C_3 : 8AB$ with a fixed component $\bar{C}$ and meet $C_3 : 8AB$ in three variable points. If B is a basis point of (7), $\bar{C}$ must meet $C_3 : 8AB$ in three variable points. Hence $\bar{C}$ is reducible to a line and (7) is $C_4 : 9A$. If B is not a basis point of (7) either (7) is $C_3 : 6A$ so that $\bar{C} = 1$ or else $\bar{C}$ meets $C_3 : 8A$ in 2 variable points. From the discussion of I_2 it is seen that $\bar{C}$ is $C_1 : A$, $C_3 : 7A$ or $C_6 : 8A^2$. For the pencil $C_9 : 8A^3B^3$ we readily find for (7) $C_9 : 8A^3B^2$, $C_{12} : 8A^4B^3$, or $C_{15} : 8A^5B^4$.

The corresponding families can readily be formed by remembering that from (9) $c_1C_1u + c_2C_2u + c_3v$ is linear in the curves of (1) with coefficients of order n in the curves of (2). Hence we find the following:

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* K. Doehlmann, *Geometrische Transformationen*, Vol. 2, p. 170.

† G. H. Halphen, *Bulletin de la Société Mathématique*, Vol. 10 (1882), pp. 162-172.

Families of Involutions of Order 3.

	Pencil	*Linear System*	*Family*
(1)	$C_3 : 6A3B$	$C_3 : 6A$	$C_{3n} : 6A^n3B^{n-1}\ (6n-6)C$
(2)	$C_3 : 9A$	$C_4 : 9A$	$C_{3n+1} : 9A^n\ (6n-2)B$
(3)	$C_3 : A^27BC$	$C_4 : A^27B$	$C_{3n+1} : A^{n+1}7B^nC^{n-1}\ (6n-4)D$
(4)	$C_3 : 7ABC$	$C_6 : 7A^2B$	$C_{3n} : 7A^nB^{n-1}C^{n-2}\ (6n-8)D$
(5)	$C_3 : 8AB$	$C_9 : 8A^3$	$C_{3n} : 8A^nB^{n-3}\ (6n-12)C$
(6)	$C_9 : 8A^3B^3$	$C_9 : 8A^3B^2$	$C_{3n} : 8A^nB^{n-1}\ (2n-4)C$
		$C_{12} : 8A^4B^3$	
		$C_{15} : 8A^5B^4$	

In each case the smallest possible n gives the linear system from which the family was derived and corresponds to the net of plane sections of F_N^3 through a point on l, the simple basis points being the images of this point. For the families, the images of points on C_{2n} and F_N^3 but not on l are simple basis points. The corresponding F_N^3's are

(1) $F_3 : r = 3$, l not on F_3

(2) $F_7 : r = 5 : l^4, l_1^3, l_2^3$ meeting l

(3) $F_5 : r = 4 : l^2, l_1^3$, meeting l

(4) $F_7 : r = 5 : l^4, l_1^3, l_2^3$ concurrent

(5) $F_9 : r = 6 : l^6, l_1^3, l_2^3$ concurrent.

8. *Application to I_4.* With the pencil $C_3 : 8AB$ and $t = 4$ we first deduce from I_3, by freeing one point, the linear systems

$$C_3 : 5A, \quad C_4 : 8A, \quad C_4 : A^26B, \quad C_6 : 7A^2.$$

If B is to be a basis point of (1) then $C_3 : 8AB$ must meet $\bar{C}$ in 4 variable points. Hence $\bar{C} = C_2 : 2A$. If B is not a basis point of (1) $C_3 : 8AB$ must meet $\bar{C}$ in 3 variable points, that is, from I_3

$$\bar{C} = C_3 : 6A, \quad C_4 : A^27B, \quad C_6 : 7A^2B, \text{ or } C_9 : 8A^3.$$

Hence including the pencils $C_6 : 8A^2B^2$ and $C_{12} : 8A^4B^4$ we have the following:

Families of Involutions of Order 4, I_4

	Pencil	*Linear System*	*Family*
(1)	$C_3 : 5A4B$	$C_3 : 5A$	$C_{3n} : 5A^n4B^{n-1}\ (8n-8)C$
(2)	$C_3 : 8AB$	$C_4 : 8A$	$C_{3n+1} : 8A^nB^{n-1}\ (8n-4)C$
(3)	$C_3 : A6B2C$	$C_4 : A^26B$	$C_{3n+1} : A^{n+1}6B^n2C^{n-1}\ (8n-6)D$
(4)	$C_3 : 7A2B$	$C_6 : 7A^2$	$C_{3n} : 7A^n2B^{n-2}\ (8n-12)C$
(5)	$C_3 : 2A7B$	$C_5 : 2A^27B$	$C_{3n+2} : 2A^{n+1}7B^n\ (8n-2)C$
(6)	$C_3 : 6A2BC$	$C_6 : 6A^22B$	$C_{3n} : 6A^n2B^{n-1}C^{n-2}\ (8n-10)D$
(7)	$C_3 : A7BC$	$C_7 : A^37B^2$	$C_{3n+1} : A^{n+1}7B^nC^{n-2}\ (8n-8)D$
(8)	$C_3 : 7ABC$	$C_9 : 7A^3B^2$	$C_{3n} : 7A^nB^{n-1}C^{n-3}\ (8n-14)D$
(9)	$C_3 : 8AB$	$C_{12} : 8A^4$	$C_{3n} : 8A^nB^{n-4}\ (8n-20)C$
(10)	$C_6 : 8A^2B^2$	$C_9 : 8A^3B$	$C_{3n} : 8A^nB^{n-1}\ (2n-5)C$
(11)	$C_{12} : 8A^4B^4$	$C_{12} : 8A^4B^3$	
		$C_{15} : 8A^5B^4$	
		$C_{18} : 8A^6B^5$	
		$C_{21} : 8A^7B^6$	

The corresponding surfaces $F_N{}^3$ for (6) of I_3 and (10), (11) of I_4 have complex singular points which will be discussed elsewhere. It can be seen from the method here developed that if we know the linear systems for I_t genus p we can deduce the linear systems for the families of I_{t+1} genus p.

9. *Single types of I_3 not belonging to a family.*

(1) $C_4 : 13A$.* From either

$$z_1x_1 + z_2x_2 + z_3x_3 = 0, \quad z_1C_1 + z_2C_2 + z_3C_3 = 0 \text{ (cubic curves)}$$

or $C_4 : 11A2B$, the two points B being properly chosen. The transformation $y_i = C_i$, $(i = 1 \text{ to } 4)$, $C_i = 0$ being a $C_4 : 11A$, maps the plane on an $F_5{}^3$ with a double space cubic. The I_3 corresponds to cutting $F_5{}^3$ by the lines through a point on the double curve.

(2) $C_7 : A^39B^2C$ † the points $A9B$ being on a cubic, the image of the singular point of the $F_4{}^3$ determined by the transformation, $y_i = C_3 : A9B$, C_i $(i = 1, 2, 3)$, where $C_i = 0$ is a $C_4 : A^29B$, $y_4 = C_7 : A^39B^2$. The I_3 corresponds to cutting $F_4{}^3$ by lines through a point on the surface.

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* F. S. Macaulay, *Proceedings of the London Mathematical Society*, Vol. 26 (1895), p. 514.

† M. Noether, *Mathematische Annalen*, Vol. 33 (1889), p. 546.

(3) $C_7 : A^3 8 B^2 2 C D 2 E$ * the points $A8B2C$ being on a cubic the image of the singular point of the F_5^3 determined by the transformation $y_i = C_3 : A8B2C \cdot C_i$, where $C_i = 0$ is a $C_4 : A^2 8BD$, $y_4 = C_7 : A^3 8 B^2 2 CD$. The I_3 corresponds to cutting F_5^3 by lines through a point on the double conic of the surface.

(4) $C_9 : 7A^3 3B^2 C 2D$,* the points $7A3B$ being on a cubic the image of the singular point of the F_5^3 determined by the transformation $y_i = C_3 : 7A3B \cdot C_i$, where $C_i = 0$ is a $C_6 : 7A^2 3BC$, $y_4 = C_9 : 7A^3 3B^2 C$. The I_3 corresponds to cutting F_5^3 by lines through a point on the double conic.

10. *Conclusion.* The families of involutions I_4 determined from pencils of genus 2 and 3 are being investigated by Franklin G. Williams of Cornell University. When they are completed a knowledge of the linear systems of least order for $p = 5$, and 6 which are also being determined would enable one to determine the single types of I_4 not included in families. I wish to acknowledge my indebtedness to the work of D. Montesano * on Quintic Surfaces with a double conic, a double line or no double curve, but in all these cases, with point singularities such that the surfaces are rational. Many of the involutions were first found from the mapping of these surfaces as given by Montesano. While the involutions thus found are special cases of the involutions here given, an incomplete list of families of involutions I_3, I_4 can readily be found from his valuable memoirs.

* D. Montesano, *Napoli Rendiconti*, Ser. 3, Vol. 7 (1901), pp. 67-106; Ser. 3, Vol. 13 (1907), pp. 66-88.

CORRECTIONS, VOLUME 49.

H. M. GEHMAN, *Irreducible Continuous Curves.*

Page 191, line 16, change "$*$" to "$\ddagger$."

Page 196, line 2, change "*continuum*" to "*subcontinuum.*"

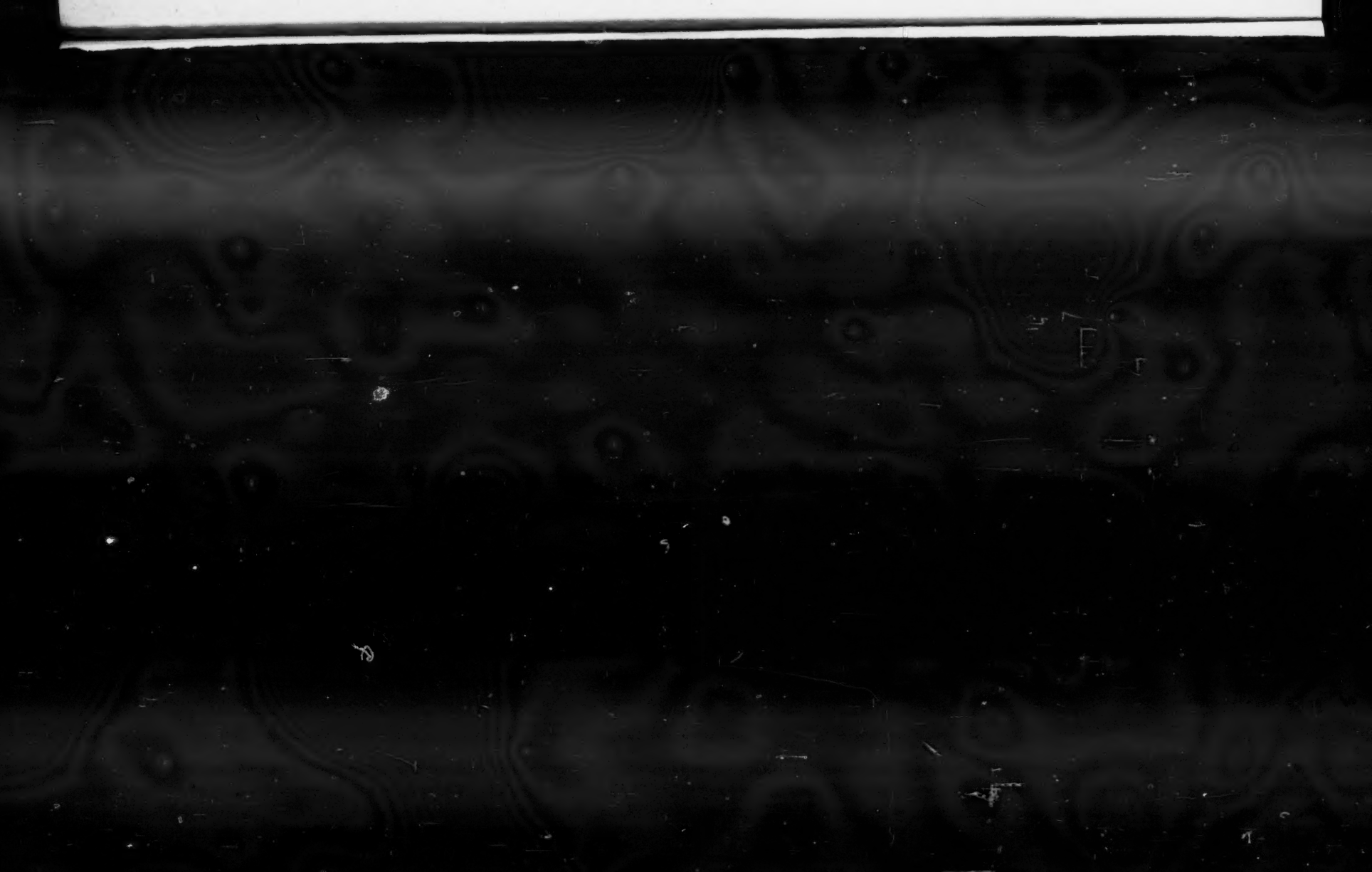
CORRECTIONS, VOLUME 50.

G. T. WHYBURN, *Concerning the Structure of a Continuous Curve.*

Page 170, line 20, change "H" to "H_2."

Page 172, line 9 from bottom, change "E" to "E_2."

Page 184, the third footnote should include the following reference: W. L. Ayres, "Concerning Continuous Curves and Correspondences," *Annals of Mathematics,* Vol. 28 (1927), pp. 396-418, Theorem 5.



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